INTRODUCTION TO CONFORMAL GEOMETRY

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Contents

Some notations:

(1) $\Sigma^k(V) \subset \otimes^k(V)$ denotes the subspace of symmetric tensors.
(2) $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

A brief introduction

Conformal geometry is an important tool to study curvatures. The basic topic of conformal geometry is the conformal deformation of curvature tensors.

Except the standard curvature tensors (Riemannian curvature tensor, Ricci curvature tensor and scalar curvature tensor), we put forward two new curvature tensors: Schouten tensor and Weyl tensor. One the one hand, they naturally come from the decomposition of Riemannian curvature tensor with respect to the trace operator and determine the properties of curvatures. On the other hand, they have relatively better properties under conformal transformations.

In conformal geometry we know the basic fact: a manifold is locally conformally flat iff the Weyl tensor equals to zero (in dimension $n > 3$). We also classify the properties of Schouten tensor and Weyl tensor and the study of locally conformal flat manifolds into the category of conformal geometry.

We put forward two applications.

(1) First, we generalize the scalar curvature via Schouten tensor, and give the curvature estimate on locally conformally flat manifolds.
(2) Second, based on the transformation laws of curvature tensors under conformal transformations, we study the prescribed curvature problem via the theory of elliptic partial differential equations.

In addition, we point out that due to space constraints, we only introduce the 2-dimensional case in detail for the second application. In fact, Weyl tensor still plays an important role in high-dimensional cases, which we do not introduce in detail.

1. Preliminaries

A motivating problem (corollary 1.16):

Problem 1.1. *Let (M, g) be a 3-dimensional Riemannian manifold. If g is an Einstein metric, then M has constant sectional curvatures.*

1.A. Algebraic curvature tensors — trace operator, Kulkarni-Nomizu product.

To study the curvature tensors, we put forward the concept of algebraic curvature tensor on V:

Definition 1.2 (Algebraic curvature tensor)**.** *Let V be an n-dimensional real vector space. Then we call $T \in \otimes^4(V^*)$ an **algebraic curvature tensor** on V if it satisfies:*

(1) $T(x, y, z, w) = -T(y, x, z, w) = -T(x, y, w, z) = T(z, w, x, y)$ for all $x, y, z, w \in V$;
(2) $T(x, y, z, w) + T(y, z, x, w) + T(z, x, y, w) = 0$ for all $x, y, z, w \in V$.

We denote the space of algebraic curvature tensors on V by $\mathcal{R}(V^*)$.

Remark 1.3. The Riemannian curvature tensors satisfy the differential property in addition (see Bianchi second identify [Pet16, proposition 3.1.1]). Here we just use the pointwise algebraic properties of Riemannian curvature tensors.

In the next we introduce some basic conclusions of linear algebra. First, we note that $\mathcal{R}(V^*)$ is a linear subspace of $\otimes^4(V^*)$, and its dimension is computed as follows.

Proposition 1.4. *Let V be an n-dimensional real vector space. Then*

$$\dim \mathcal{R}(V^*) = \frac{n^2(n^2-1)}{12}$$

Proof. See [Lee18, proposition 7.21]. □

Second, we introduce the trace operator $\mathrm{tr}_g : \mathcal{R}(V^*) \to \Sigma^2(V^*)$, and try to study $\mathcal{R}(V^*)$ via this linear map.

This is just like the process that we derive the Ricci curvature tensor via a Riemannian curvature tensor. For any $g \in \Sigma^2(V^*)$ which is nondegenerate (not necessarily positive definite), we define a map

$$\mathrm{tr}_g : \mathcal{R}(V^*) \to \Sigma^2(V^*), \quad T \mapsto g^{il} T_{ijkl}. \tag{1.1}$$

Clearly this is well-defined. It is natural to wonder whether this operator is surjective and what its kernel is,[1] as a way of asking how much of information contained in the Riemannian curvature tensor is captured by the Ricci tensor.

One way to try to answer the question is to attempt to construct a right inverse for the trace operator.[2] A natural right inverse operator is induced by the Kulkarni-Nomizu product, which is a natural product operator that yields algebraic curvature operators.

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[1] In other words, we want to decomposes T orthogonally into the traceless part and its orthogonal.
[2] I.e. a linear map $G : \Sigma^2(V^*) \to \mathcal{R}(V^*)$ such that $\mathrm{tr}_g \circ G = \mathrm{id}$.

Definition 1.5 (Kulkarni-Nomizu product). *Let V be a real vector space. For $h, k \in \Sigma^2(V^*)$, we define the* ***Kulkarni-Nomizu product*** *of h and k by the following formula*

$$\begin{aligned}(h \owedge k)(v_1, v_2, v_3, v_4) &= \frac{1}{2}\left(h(v_1, v_4) \cdot k(v_2, v_3) + h(v_2, v_3) \cdot k(v_1, v_4)\right) \\ &\quad - \frac{1}{2}\left(h(v_1, v_3) \cdot k(v_2, v_4) + h(v_2, v_4) \cdot k(v_1, v_3)\right)\end{aligned}$$

The factor $1/2$ is not used consistently in the literature, but is convenient when $h = k$.

Proposition 1.6. *Let V be an n-dimensional real vector space, and let $g \in \Sigma^2(V^*)$ be nondegenerate. Then for $h, k \in \Sigma^2(V^*)$, we know:*

(1) $h \owedge k$ is an algebraic curvature tensor;
(2) $h \owedge k = k \owedge h$;
(3) $2\mathrm{tr}_g(h \owedge g) = (n-2)h + (\mathrm{tr}_g h)\, g$;
(4) $\mathrm{tr}_g(g \owedge g) = (n-1)g$;
(5) $\langle T, h \owedge g\rangle_g = 2\left\langle \mathrm{tr}_g T, h\right\rangle_g$;
(6) In case g is positive definite, $|g \owedge h|_g^2 = (n-2)|h|_g^2 + (\mathrm{tr}_g h)^2$.

Proof. Points (1) and (2) are trivial. Note that

$$\begin{aligned}2\mathrm{tr}_g(h \owedge g)_{jk} &= g^{il}\left(h_{il}g_{jk} + h_{jk}g_{il} - h_{ik}g_{jl} - h_{jl}g_{ik}\right) \\ &= h_i{}^i g_{jk} + nh_{jk} - h_{jk} - h_{jk} = h_i{}^i g_{jk} + (n-2)h_{jk}.\end{aligned}$$

Hence we get point (3). Point (4) follows from point (3) immediately. Also note that

$$\begin{aligned}2\langle T, h \owedge g\rangle_g &= T^{ijkl}\left(h_{il}g_{jk} + h_{jk}g_{il} - h_{ik}g_{jl} - h_{jl}g_{ik}\right) \\ &= T^{ijkl}h_{il}g_{jk} + T^{ijkl}h_{jk}g_{il} - T^{ijkl}h_{ik}g_{jl} - T^{ijkl}h_{jl}g_{ik} \\ &= T^{jilk}h_{il}g_{jk} + T^{ijkl}h_{jk}g_{il} + T^{jikl}h_{ik}g_{jl} + T^{ijlk}h_{jl}g_{ik} = 4\left\langle \mathrm{tr}_g T, h\right\rangle_g.\end{aligned}$$

Hence we get point (5). Then it follows from the preceding points that

$$\begin{aligned}|g \owedge h|_g^2 &= \langle g \owedge h, g \owedge h\rangle_g = 2\left\langle \mathrm{tr}_g(g \owedge h), h\right\rangle_g \\ &= \left\langle (n-2)h + (\mathrm{tr}_g h)\, g, h\right\rangle_g = (n-2)|h|_g^2 + (\mathrm{tr}_g h)^2\end{aligned}$$

where we use the fact that $\langle g, h\rangle_g = \mathrm{tr}_g h$. □

Proposition 1.7. *Let V be an n-dimensional real vector space with $n \geq 3$, and let $g \in \Sigma^2(V^*)$ be nondegenerate. We define a linear map*

$$G : \Sigma^2(V^*) \to \mathcal{R}(V^*), \quad h \mapsto \left(\frac{2}{n-2}h - \frac{\mathrm{tr}_g h}{(n-1)(n-2)}\, g\right) \owedge g.$$

Then G is a right inverse for tr_g. Moreover, we have

$$\mathrm{im}\,(G) = \left(\ker\left(\mathrm{tr}_g\right)\right)^{\perp}.$$

Proof. The fact that $\mathrm{tr}_g \circ G = \mathrm{id}$ follows from proposition 1.6 (3) (4), i.e. G is a right inverse for tr_g. Via simple linear algebra, it follows that that

$$\dim \mathrm{im}\,(G) = \dim\left(\ker\left(\mathrm{tr}_g\right)\right)^{\perp}.$$

On the other hand, proposition 1.6 (5) yields that

$$\operatorname{im}(G) \subset \left(\ker\left(\operatorname{tr}_g\right)\right)^{\perp}.$$

Therefore we get the conclusion by dimensionality. □

Remark 1.8. We hence get an orthogonal decomposition

$$\mathcal{R}(V^*) = \ker\left(\operatorname{tr}_g\right) \oplus \operatorname{im}(G)$$

These results lead to some important simplifications in low dimensions.

Corollary 1.9. *Let V be an n-dimensional real vector space, and let $g \in \Sigma^2(V^*)$ be nondegenerate.*

(1) If $n = 0$ or $n = 1$, then $\mathcal{R}(V^*) = \{0\}$.
(2) If $n = 2$, then $\mathcal{R}(V^*) = \operatorname{span}\{g \owedge g\}$.
(3) If $n = 3$, then $\dim \mathcal{R}(V^*) = 6$, and $G : \Sigma^2(V^*) \to \mathcal{R}(V^*)$ is an isomorphism.

Proof. The dimensional results follow immediately from proposition 1.4.

In the case $n = 2$, proposition 1.6 (4) implies that $\operatorname{tr}_g(g \owedge g) = g \neq 0$. Therefore, $g \owedge g \neq 0$ and hence spans the 1-dimensional space $\mathcal{R}(V^*)$.

In the case $n = 3$, proposition 1.7 implies that G is injective (using $\operatorname{tr}_g \circ G = \mathrm{id}$). Therefore, G is an isomorphism by dimensionality. □

Remark 1.10. Corollary 1.9 (3) implies that the Riemannian curvature tensor will be determined by the Ricci tensor in dimension 3. In fact, in dimension 3, $\operatorname{tr}_g = G^{-1}$, and hence

$$R = G\left(\operatorname{tr}_g R\right) = G(\operatorname{Ric}). \tag{1.2}$$

1.B. **Weyl tensor and Schouten tensor.** Now we apply the conclusions of subsection 1.A to (pseudo-)Riemannian manifolds. We focus on the case that $h = \operatorname{Ric}$.

Definition 1.11 (Weyl tensor and Schouten tensor)**.** *Let g be a Riemannian or pseudo-Riemannian metric. Define the **Schouten tensor** of g by*

$$P = \frac{2}{n-2}\operatorname{Ric} - \frac{\operatorname{scal}}{(n-1)(n-2)} \cdot g \tag{1.3}$$

*and define the **Weyl tensor** of g by*

$$W = R - P \owedge g = R - \frac{2}{n-2}\operatorname{Ric} \owedge g + \frac{\operatorname{scal}}{(n-1)(n-2)} \cdot g \owedge g. \tag{1.4}$$

Remark 1.12. Clearly, we have $G(\operatorname{Ric}) = P \owedge g$.

Proposition 1.13. *For every (pseudo-)Riemannian manifold (M, g) of dimension $n \geq 3$,*

$$R = W + P \owedge g \tag{1.5}$$

is the orthogonal decomposition of R corresponding to $\mathcal{R}(T_p^*M) = \ker\left(\operatorname{tr}_g\right) \oplus \left(\ker\left(\operatorname{tr}_g\right)\right)^{\perp}$. (This implies that the trace of Weyl tensor is zero.)

Proof. It follows directly from proposition 1.7. □

Proposition 1.14. *On every (pseudo-)Riemannian manifold (M, g) of dimension 2, the Riemannian and Ricci tensors are determined by the scalar curvature as follows:*

$$R = \frac{\text{scal}}{2} g \owedge g \quad \textit{and} \quad \text{Ric} = \frac{\text{scal}}{2} g.$$

Proof. By corollary 1.9 (2), there exists $f \in C^\infty(M)$ such that $R = fg \owedge g$. Taking traces, we get via proposition 1.6 (4) that $\text{Ric} = fg$, and then $\text{scal} = \text{tr}(\text{Ric}) = 2f$. Done. □

Proposition 1.15. *On every (pseudo-)Riemannian manifold (M, g) of dimension 3, the Weyl tensor is zero, and Riemannian curvature tensor is determined by the Ricci tensor via the formula*

$$R = P \owedge g = 2\text{Ric} \owedge g - \frac{\text{scal}}{2} g \owedge g.$$

Proof. Corollary 1.9 shows that $G : \Sigma^2(V^*) \to \mathcal{R}(V^*)$ is an isomorphism in dimension 3. Since $\text{tr}_g \circ G = \text{id}$, we know $\text{tr}_g = G^{-1}$ is also an isomorphism. Bacause $\text{tr}_g W = 0$ by proposition 1.13, it follows that $W = 0$. The second assertion follows from (1.2). □

Corollary 1.16. *Let (M, g) be a 3-dimensional Riemannian manifold. If g is an Einstein metric, then M has constant sectional curvatures.*

Proof. The conclusion follows from proposition 1.15 and Schur theorem 6.1. □

Moreover, using the traceless Ricci tensor, we can further decompose the Riemannian curvature tensor.

Proposition 1.17. *Let (M, g) be a (pseudo-)Riemannian manifold of dimension $n \geq 3$. Then the $(0, 4)$-curvature tensor of g has the following orthogonal decomposition:*

$$R = W + \frac{2}{n-2} \text{Ric}^\circ \owedge g + \frac{\text{scal}}{n(n-1)} g \owedge g.$$

Therefore, in the Riemannian case,

$$\begin{aligned} |R|_g^2 &= |W|_g^2 + \frac{4}{(n-2)^2} \left|\text{Ric}^\circ \owedge g\right|_g^2 + \frac{\text{scal}^2}{n^2(n-1)^2} |g \owedge g|_g^2 \\ &= |W|_g^2 + \frac{4}{n-2} \left|\text{Ric}^\circ\right|_g^2 + \frac{2}{n(n-2)} \text{scal}^2. \end{aligned}$$

Proof. It follows from the definition and proposition 1.6. □

1.C. **Curvatures of conformally related metrics.** By formula (1.5), we can reduce the analysis of Riemannian curvature R to the analysis of Weyl tensor W and Schouten tensor P.

An important property of Weyl tensor W is its transformation law under conformal changes of metric. Moreover, we can use W to judge whether the manifold is locally conformally flat, which explains the geometric significance of the Weyl tensor.

Definition 1.18. *Two Riemannian or pseudo-Riemannian metrics on the same manifold are said to be* ***conformal*** *to each other if one is a positive function times the other.*

Proposition 1.19. *Let (M,g) be a (pseudo-)Riemannian n-manifold (with or without boundary), and let $\widetilde{g} = e^{2f}g$ be any metric conformal to g. If ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections of g and $\widetilde{g}$ respectively, then*

$$\widetilde{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X - \langle X, Y\rangle \cdot \nabla f. \tag{1.6}$$

In any local coordinates, the Christoffel symbols of the two connections are related by

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + f_{;i}\delta_j^k + f_{;j}\delta_i^k - g^{kl}f_{;l}g_{ij}. \tag{1.7}$$

Proof. Recall that

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

Then we know

$$\begin{aligned}
\widetilde{\Gamma}_{ij}^k &= \frac{1}{2}e^{-2f}g^{kl}\left(\partial_i\left(e^{2f}g_{jl}\right) + \partial_j\left(e^{2f}g_{il}\right) - \partial_l\left(e^{2f}g_{ij}\right)\right) \\
&= \frac{1}{2}e^{-2f}g^{kl}\left(e^{2f}\partial_i g_{jl} + e^{2f}\partial_j g_{il} - e^{2f}\partial_l g_{ij}\right) \\
&\quad + \frac{1}{2}e^{-2f}g^{kl}\left(2e^{2f}f_{;i}g_{jl} + 2e^{2f}f_{;j}g_{il} - 2e^{2f}f_{;l}g_{ij}\right) \\
&= \Gamma_{ij}^k + f_{;i}g^{kl}g_{jl} + f_{;j}g^{kl}g_{il} - f_{;l}g^{kl}g_{ij} \\
&= \Gamma_{ij}^k + f_{;i}\delta_j^k + f_{;j}\delta_i^k - g^{kl}f_{;l}g_{ij}.
\end{aligned}$$

Hence we get formula (1.7). Clearly, formula (1.6) is a straightforward computation using formula (1.7) in coordinates. We are done. □

Corollary 1.20 (Laplacian on functions)**.** *Let (M,g) be a (pseudo-)Riemannian manifold (with or without boundary), and let $\widetilde{g} = e^{2f}g$ be any metric conformal to g. Then for smooth function ϕ, it holds that*

$$\widetilde{\Delta}\phi = e^{-2f}\left(\Delta\phi - (n-2)\langle\nabla f, \nabla\phi\rangle_g\right).$$

Proof. Note that the Laplacian on functions can be expressed as

$$\Delta\phi = g^{jk}\frac{\partial^2\phi}{\partial x^j\partial x^k} - g^{jk}\Gamma_{jk}^l\frac{\partial\phi}{\partial x^l}.$$

Then the conclusion easily follows from (1.7). □

Proposition 1.21. *Let (M,g) be a (pseudo-)Riemannian n-manifold (with or without boundary), and let $\widetilde{g} = e^{2f}g$ be any metric conformal to g. In the Riemannian case, the curvature tensors of $\widetilde{g}$ (represented with tildes) are related to those of g by the following formulas:*

$$\widetilde{R} = e^{2f}\left(R - 2\mathrm{Hess}\, f \owedge g + 2(df\otimes df)\owedge g - |\nabla f|_g^2 \cdot g \owedge g\right), \tag{1.8}$$

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (n-2)\mathrm{Hess}\, f + (n-2)(df\otimes df) - \left(\Delta f + (n-2)|\nabla f|_g^2\right)g, \tag{1.9}$$

$$\widetilde{\mathrm{scal}} = e^{-2f}\left(\mathrm{scal} - 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|_g^2\right). \tag{1.10}$$

If in addition $n \geq 3$, then

$$\widetilde{P} = P - 2\mathrm{Hess}\, f + 2df\otimes df - |\nabla f|_g^2\cdot g, \tag{1.11}$$

$$\widetilde{W} = e^{2f} W. \tag{1.12}$$

In the pseudo-Riemannian case, the same formula hold with each occurrence of $|\nabla f|_g^2$ *replaced by* $\langle \nabla f, \nabla f \rangle_g$.

Proof. Recall that

$$R_{ijkl} = g_{lm}\left(\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^r \Gamma_{ir}^m - \Gamma_{ik}^r \Gamma_{jr}^m\right).$$

Let (x^i) be normal coordinates centered at p. Then at p we have[3]

$$\begin{aligned} f_{;i;j} &= \partial_j \partial_i f, \\ \widetilde{\Gamma}_{ij}^k &= f_{;i}\delta_j^k + f_{;j}\delta_i^k - g^{kl} f_{;l} g_{ij}, \\ \partial_m \widetilde{\Gamma}_{ij}^k &= \partial_m \Gamma_{ij}^k + f_{;i;m}\delta_j^k + f_{;j;m}\delta_i^k - g^{kl} f_{;l;m} g_{ij}. \end{aligned}$$

Then we have

$$\begin{aligned} \widetilde{R}_{ijkl} &= \widetilde{g}_{lm}\left(\partial_i \widetilde{\Gamma}_{jk}^m - \partial_j \widetilde{\Gamma}_{ik}^m + \widetilde{\Gamma}_{jk}^r \widetilde{\Gamma}_{ir}^m - \widetilde{\Gamma}_{ik}^r \widetilde{\Gamma}_{jr}^m\right) \\ &= e^{2f}\Big(R_{ijkl} - (f_{;i;l} g_{jk} + f_{;j;k} g_{il} - f_{;i;k} g_{jl} - f_{;j;l} g_{ik}) \\ &\quad + (f_{;i} f_{;l} g_{jk} + f_{;j} f_{;k} g_{il} - f_{;i} f_{;k} g_{jl} - f_{;j} f_{;l} g_{ik}) \\ &\quad - g^{mr} f_{;m} f_{;r}\left(g_{il} g_{jk} - g_{ik} g_{jl}\right)\Big) \end{aligned}$$

which is the coordinates version of (1.8). Then the rest of this proposition follows from proposition 1.6 and formulas (1.3) and (1.4). □

In the next we begins to explain the geometric significance of the Weyl tensor.

Definition 1.22. *A Riemannian manifold is said to be* ***locally conformally flat*** *if every point has a neighborhood that is conformally equivalent to an open subset of Euclidean space.*

Similarly, a pseudo-Riemannian manifold is said to be locally conformally flat if every point has a neighborhood that is conformally equivalent to an open subset of pseudo-Euclidean space.

Corollary 1.23. *Suppose that* (M, g) *is a (pseudo-)Riemannian manifold of dimension* $n \geq 3$. *If* g *is locally conformally flat, then its Weyl tensor vanishes identically.*

Proof. It follows from proposition 1.21 directly.[4] □

In fact, in dimension $n \geq 4$, $W = 0$ is also a sufficient condition. But in dimension 3, as we showed in proposition 1.15, we always have $W = 0$. So to understand that case, we must introduce one more tensor field.

Definition 1.24. *On a (pseudo-)Riemannian manifold, the* ***Cotton tensor*** C *is defined by*

$$2C = -DP \quad \text{i.e.} \quad 2C_{ijk} = P_{ij;k} - P_{ik;j}$$

where D *is the exterior covariant derivative, i.e.*

$$(DT)(X, Y, Z) = -(\nabla_Z T)(X, Y) + (\nabla_Y T)(X, Z) \quad \textit{for any } (0,2)\textit{-tensor } T.$$

[3]Covariant derivative and directional derivative are mixed in this calculation. It will be convenient to unify them with normal coordinates.

[4]Note that for (pseudo-)Euclidean spaces $R = W = 0$ and $P = 0$.

Proposition 1.25. *Let (M, g) be a (pseudo-)Riemannian manifold of dimension $n \geq 3$, and let W and C denote its Weyl and Cotton tensors respectively. Then*

$$C_{1,2}(\nabla W) = (n-3)C.$$

Proof. The equation $W = R - P \owedge g$ yields that

$$2W_{ijkl} = 2R_{ijkl} - P_{il}g_{jk} - P_{jk}g_{il} + P_{ik}g_{jl} + P_{jl}g_{ik},$$

and hence

$$2W_{ijkl}{}^{;i} = 2R_{ijkl}{}^{;i} - P_{il}{}^{;i}g_{jk} - P_{jk}{}^{;i}g_{il} + P_{ik}{}^{;i}g_{jl} + P_{jl}{}^{;i}g_{ik}.$$

Note that the second Bianchi identify yields that

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0 \overset{\cdot g^{im}}{\Longrightarrow} R_{ijkl}{}^{;i} = R_{jk;l} - R_{jl;k} \overset{\cdot g^{jk}}{\Longrightarrow} R_{il}{}^{;i} = \frac{1}{2}\mathrm{scal}_{;l}.$$

Then by formula (1.3), we know

$$P_{ij} = \frac{2}{n-2}R_{ij} - \frac{\mathrm{scal}}{(n-1)(n-2)}g_{ij} \Longrightarrow P_{il}{}^{;i} = \frac{\mathrm{scal}_{;l}}{n-1}$$

It follows that

$$\begin{aligned} 2W_{ijkl}{}^{;i} &= 2R_{ijkl}{}^{;i} - P_{il}{}^{;i}g_{jk} - P_{jk}{}^{;i}g_{il} + P_{ik}{}^{;i}g_{jl} + P_{jl}{}^{;i}g_{ik} \\ &= 2R_{jk;l} - 2R_{jl;k} - \frac{\mathrm{scal}_{;l}}{n-1}g_{jk} - P_{jk;l} + \frac{\mathrm{scal}_{;k}}{n-1}g_{jl} + P_{jl;k} \\ &= (n-2)\left(P_{jk;l} - P_{jl;k}\right) - P_{jk;l} + P_{jl;k} = 2(n-3)C_{jkl}. \end{aligned}$$

Hence we get the conclusion. □

Remark 1.26. Basic idea of computation: we first reduce it to an equation about curvature tensors R, P, W, and then use the special properties of curvature tensors.

Proposition 1.27. *Let (M, g) be a (pseudo-)Riemannian manifold of dimension $n \geq 3$, and let $\widetilde{g} = e^{2f}g$ for some $f \in C^\infty(M)$. If C and $\widetilde{C}$ denote the Cotton tensors of g and $\widetilde{g}$ respectively, then*

$$\widetilde{C} = C + \iota_{\nabla f}W \quad \text{i.e.} \quad \widetilde{C}_{ijk} = C_{ijk} + f^{;l}W_{lijk}.$$

Proof. Formula (1.11) yields that

$$\widetilde{P}_{ij} = P_{ij} - 2f_{;i;j} + 2f_{;i}f_{;j} - f^{;m}f_{;m}g_{ij}, \tag{1.13}$$

and hence

$$\begin{aligned} \widetilde{P}_{ij;k} &= P_{ij;k} - 2f_{;i;j;k} + 2f_{;i;k}f_{;j} + 2f_{;i}f_{;j;k} - f^{;m}{}_{;k}f_{;m}g_{ij} - f^{;m}f_{;m;k}g_{ij} \\ &= P_{ij;k} - 2f_{;i;j;k} + 2f_{;i;k}f_{;j} + 2f_{;i}f_{;j;k} - 2f^{;m}f_{;m;k}g_{ij}. \end{aligned}$$

It follows that

$$\begin{aligned} \widetilde{P}_{ij;k} - \widetilde{P}_{ik;j} = {} & \left(P_{ij;k} - P_{ik;j}\right) - 2\left(f_{;i;j;k} - f_{;i;k;j}\right) \\ & + 2\left(f_{;i;k}f_{;j} - f_{;i;j}f_{;k}\right) - 2\left(f^{;m}f_{;m;k}g_{ij} - f^{;m}f_{;m;j}g_{ik}\right). \end{aligned} \tag{1.14}$$

Moreover, by formula (1.7) we know

$$\begin{aligned}
\widetilde{P}_{ij;\widetilde{k}} &= (\widetilde{\nabla}_{\partial_k}\widetilde{P})(\partial_i,\partial_j) = \partial_k\widetilde{P}_{ij} - \widetilde{P}(\widetilde{\nabla}_{\partial_k}\partial_i,\partial_j) - \widetilde{P}(\partial_i,\widetilde{\nabla}_{\partial_k}\partial_j) \\
&= \partial_k\widetilde{P}_{ij} - \widetilde{P}(\widetilde{\Gamma}^l_{ki}\partial_l,\partial_j) - \widetilde{P}\left(\partial_i,\widetilde{\Gamma}^l_{kj}\partial_l\right) \\
&= \partial_k\widetilde{P}_{ij} - \widetilde{P}(\Gamma^l_{ki}\partial_l,\partial_j) - \widetilde{P}\left(\partial_i,\Gamma^l_{kj}\partial_l\right) \\
&\quad -(f_{;k}\delta^l_i + f_{;i}\delta^l_k - g^{ls}f_{;s}g_{ki})\widetilde{P}_{lj} - \left(f_{;k}\delta^l_j + f_{;j}\delta^l_k - g^{ls}f_{;s}g_{kj}\right)\widetilde{P}_{il} \\
&= \widetilde{P}_{ij;k} - (f_{;k}\widetilde{P}_{ij} + f_{;i}\widetilde{P}_{kj} - f^{;l}g_{ki}\widetilde{P}_{lj} + f_{;k}\widetilde{P}_{ij} + f_{;j}\widetilde{P}_{ik} - f^{;l}g_{kj}\widetilde{P}_{il})
\end{aligned}$$

and hence by the symmetry of $\widetilde{P}$ and g we know

$$\widetilde{P}_{ij;\widetilde{k}} - \widetilde{P}_{ik;\widetilde{j}} = \widetilde{P}_{ij;k} - \widetilde{P}_{ik;j} - 2\,(\widetilde{P} \owedge g)_{sijk} f^{;s}. \tag{1.15}$$

Hence by formulas (1.14) and (1.15) we know

$$\begin{aligned}
\widetilde{C}_{ijk} &= C_{ijk} - (f_{;i;j;k} - f_{;i;k;j}) + (f_{;i;k}f_{;j} - f_{;i;j}f_{;k}) \\
&\quad - (f^{;m}f_{;m;k}g_{ij} - f^{;m}f_{;m;j}g_{ik}) - (\widetilde{P} \owedge g)_{sijk} f^{;s}.
\end{aligned}$$

In the next we kill the high-order terms of f.[5] Specifically, by Ricci identity 6.15 we know

$$f_{;i;j;k} - f_{;i;k;j} = -R_{kji}{}^{s} f_{;s}$$

and by (1.13) we know

$$2f_{;i;j} = P_{ij} - \widetilde{P}_{ij} + 2f_{;i}f_{;j} - f^{;s}f_{;s}g_{ij} \quad \forall i,j.$$

It follows that

$$\begin{aligned}
\widetilde{C}_{ijk} &= C_{ijk} - (f_{;i;j;k} - f_{;i;k;j}) + (f_{;i;k}f_{;j} - f_{;i;j}f_{;k}) \\
&\quad - (f^{;m}f_{;m;k}g_{ij} - f^{;m}f_{;m;j}g_{ik}) - (\widetilde{P} \owedge g)_{sijk} f^{;s} \\
&= C_{ijk} + R_{kji}{}^{s} f_{;s} - (\widetilde{P} \owedge g)_{sijk} f^{;s} \\
&\quad + \frac{1}{2}(P_{ik} - \widetilde{P}_{ik} + 2f_{;i}f_{;k} - f^{;s}f_{;s}g_{ik}) f_{;j} \\
&\quad - \frac{1}{2}(P_{ij} - \widetilde{P}_{ij} + 2f_{;i}f_{;j} - f^{;s}f_{;s}g_{ij}) f_{;k} \\
&\quad - \frac{1}{2}f^{;m}g_{ij}(P_{mk} - \widetilde{P}_{mk} + 2f_{;m}f_{;k} - f^{;s}f_{;s}g_{mk}) \\
&\quad + \frac{1}{2}f^{;m}g_{ik}(P_{mj} - \widetilde{P}_{mj} + 2f_{;m}f_{;j} - f^{;s}f_{;s}g_{mj}) \\
&= C_{ijk} + R_{sijk}f^{;s} - (\widetilde{P} \owedge g)_{sijk} f^{;s} - (P \owedge g)_{sijk} f^{;s} + (\widetilde{P} \owedge g)_{sijk} f^{;s} \\
&= C_{ijk} + W_{sijk} f^{;s}
\end{aligned}$$

We are done. ∎

Remark 1.28. One can refer to [Gre, Conformal Transformation of the Cotton Tensor] or [Gar, theorem 4.3.1] for new proofs.

Corollary 1.29. *C is a conformally invariance in dimension* 3.

[5] The basic idea for simplification is to kill the high-order terms, and then reduce our problem to the property of curvature tensors. Seeing from the result we also know that the high-order terms will cancel.

Proof. In dimension 3, proposition 1.15 shows that $W = 0$, and hence the conclusion follows from proposition 1.27. □

Corollary 1.30. *If (M, g) is a locally conformally flat 3-manifold, then the Cotton tensor of g vanishes identically.*

Proof. Note that for (pseudo-)Euclidean spaces $P = 0$ (and $R = W = 0$). Then the conclusion follows from proposition 1.27. □

The real significance of the Weyl and Cotton tensors is explained by the following important theorem.

Theorem 1.31 (Weyl-Schouten)**.** *Let (M, g) be a (pseudo-)Riemannian manifold of dimension $n \geq 3$.*

(1) If $n \geq 4$, then (M, g) is locally conformally flat iff its Weyl tensor is identically zero.
(2) If $n = 3$, then (M, g) is locally conformally flat iff its Cotton tensor is identically zero.

Proof. The necessity of each condition was proved in corollaries 1.23 and 1.30. To prove sufficiency, suppose (M, g) satisfies the hypothesis appropriate to its dimension.

First, we note that $W = 0$ and $C = 0$ by propositions 1.15 and 1.25. Moreover, by formula (1.12), every metric $\widetilde{g} = e^{2f} g$ conformal to g also has zero Weyl tensor, and hence its curvature tensor is $\widetilde{R} = \widetilde{P} \owedge \widetilde{g}$.

Second, we prove that in a neighborhood of each point, the function f can be chosen to make $\widetilde{P} = 0$, which completes the proof by proposition 6.21. From formula (1.11), it follows that $\widetilde{P} = 0$ iff

$$P - 2\mathrm{Hess}\, f + 2df \otimes df - \langle \nabla f, \nabla f \rangle \cdot g = 0. \tag{1.16}$$

To **locally** solve the above **second order PDEs** (1.16), our idea is as follows:

(1) Find a solution with df substituted by ω. (**Then it's reduced to a first order PDEs, and we may be able to apply the Frobenius theorem** (see section 7).)
(2) Show that $d\omega = 0$; then by Poincaré lemma we can find f locally.

Let $A : T^*M \to \otimes^2 T^*M$ be a smooth map given by

$$A(\omega) = \frac{P}{2} + \omega \otimes \omega - \frac{1}{2} \langle \omega, \omega \rangle \cdot g \quad \forall \omega \in \Gamma(M, T^*M).$$

By example 7.17, there exists a local solution ω to the following first order PDEs:

$$A(\omega)_{ij} = \omega_{i;j}$$

in a neighborhood of each point. Moreover, since $A(\omega)$ is symmetric, we know[6]

$$\partial_j \omega_i = \omega_{i;j} + \Gamma^s_{ij} \omega_s = \omega_{j;i} + \Gamma^s_{ij} \omega_s = \partial_i \omega_j.$$

It follows that $d\omega = 0$. Then by Poincaré lemma [Lee18, theorem 17.14], in some (possibly smaller) neighborhood of each point, there is a smooth function f with $\omega = df$; this f is the function we seek. □

[6]Here we use formula (7.13).

2. Curvature estimate and its applications

2.A. σ_k-scalar curvature.
First, we generalize the concept of scalar curvature via Schouten tensors.

Definition 2.1 (σ_k-scalar curvature)**.** *Let (M, g) be a Riemannian manifold, and let P be the Schouten tensor. Then the k-**th scalar curvature** of M is $\mathrm{tr}_k(P)$, where $\mathrm{tr}_k(P)$ is defined in definition 5.1.*

Remark 2.2. Clearly, by definition 1.11 we know

$$\mathrm{tr}_1(P) = \mathrm{tr}\,(P) = \frac{n-2}{(n-1)(n-2)}\mathrm{scal}.$$

This explains why $\mathrm{tr}_k(P)$ is called σ_k-scalar curvature.

The ideas and tools to make curvature estimates are introduced in section 5. Roughly speaking, we will add restrictions on $\mathrm{tr}_k(P)$ to derive estimates of $G_{\min}(\mathrm{Ric})$ and $G_{n,p}(P)$. These estimates have the following basic applications:

(1) the estimates of $G_{\min}(\mathrm{Ric})$ will give a lower bound of Ricci curvature (subsection 2.B);
(2) the estimates of $G_{n,p}(P)$ will lead to the vanishing theorems (subsection 2.C).

Remark 2.3. Note that $\mathrm{tr}_k(P)$'s determine the spectrum of P, which almost determine P (corollary 4.3 shows that P is determined by the spectrum and eigenvectors of $P^i{}_j$.)

Therefore, when M is conformally flat, these methods have great power, since in this case R is determined by P ($R = P \owedge g$).

Specifically, the restriction is that $g_x \in \Gamma_k^+$, $g_x \in \overline{\Gamma}_k^+$, $g \in \Gamma_k^+$, or $g \in \Gamma_k^+$

Definition 2.4. *Let (M, g) be a Riemannian manifold and $x \in M$. We say that $g_x \in \Gamma_k^+$ if*

$$\mathrm{tr}_j(P)(x) > 0 \quad \forall 1 \le j \le k,$$

and we say that $g_x \in \overline{\Gamma}_k^+$ if

$$\mathrm{tr}_j(P)(x) \ge 0 \quad \forall 1 \le j \le k.$$

Moreover, we say that $g \in \Gamma_k^+$ (resp. $g \in \overline{\Gamma}_k^+$) if $g_x \in \Gamma_k^+$ (resp. $g_x \in \overline{\Gamma}_k^+$) for all $x \in M$.

2.B. Estimates of Ricci curvature — first geometric quantity.

Theorem 2.5. *Let (M, g) be a Riemannian manifold and $x \in M$. Assume $k > 1$. If $g_x \in \Gamma_k^+$ (resp. $g_x \in \overline{\Gamma}_k^+$) for some $k \ge n/2$, this its Ricci curvature is positive (resp. non-negative) at x. Moreover, if $g \in \overline{\Gamma}_k^+$ for some $k > 1$, then*

$$\mathrm{Ric} \ge \frac{2k-n}{2n(k-1)}\mathrm{scal} \cdot g.$$

In particular, if $k \ge n/2$, then

$$\mathrm{Ric} \ge \frac{(2k-n)(n-1)}{k-1}\binom{n}{k}^{-\frac{1}{k}} \mathrm{tr}_k^{\frac{1}{k}}(P) \cdot g.$$

Proof. It follows directly from the estimate of first geometric quantity (proposition 5.6). □

In case $k \geq n/2$, if M is locally conformally flat in addition, these estimates will lead to a more precisely result about classification.

Corollary 2.6. *Let (M, g) be a compact and locally conformally flat manifold. Assume $g \in \overline{\Gamma}_k^+$ with $k \geq n/2$. Then (M, g) is conformally equivalent to either a space from or a finite quotient of Riemannian $S^{n-1}(c) \times S^1$ for some constant $c > 0$ and $k = n/2$. In particular, if $g \in \Gamma_k^+$, then (M, g) is conformally equivalent to a spherical space form.*

Proof. One can refer to [GVW02, corollary 1]. □

2.C. **Vanishing theorems — second geometric quantity.** In the next we apply the Bochner technique to differential forms on *locally conformally flat manifolds*. By subsection 6.D and Hodge theorem 6.23, the key point is to show that

$$g\left(\mathrm{Ric}(\omega), \omega\right) \geq 0, \quad \forall \omega \in \mathcal{H}^p(M)$$

where

$$\mathcal{H}^p(M) = \{\omega \in \Omega^p(M) : \Delta_H \omega = 0\}.$$

In practice, we will show that the linear operator

$$\mathrm{Ric} : \Omega^p(M) \to \Omega^p(M)$$

is non-negative (and in addition is positive at some point). Specifically, lemma 6.19 implies that

(1) if $\mathrm{Ric} : \Omega^p(M) \to \Omega^p(M)$ is non-negative, then each harmonic p-form ω is parallel;
(2) if in addition Ric is positive at some point, then $\mathcal{H} = \{0\}$.

Another key point is that the positivity of Ric is highly related to the second geometric quantity of P (in case M is locally conformally flat). Specifically,

$$\begin{aligned} G_{n,p}(P)(x) \geq 0 \quad &\Longrightarrow \quad \mathrm{Ric} : \wedge^p T_x^* M \to \wedge^p T_x^* M \text{ is non-negative} \\ G_{n,p}(P)(x) > 0 \quad &\Longrightarrow \quad \mathrm{Ric} : \wedge^p T_x^* M \to \wedge^p T_x^* M \text{ is positive} \end{aligned}$$

We will show this later.

First we simplify the Weitzenböch curvature operator Ric for differential forms.

Proposition 2.7. *Let (E_i) be a local orthonormal frame of TM and let (E^i) be its dual. For $\omega \in \Omega^*(M)$, we have*

$$\mathrm{Ric}(\omega) = \sum E^j \wedge i(E_l) R(E_l, E_j)\omega. \tag{2.1}$$

Proof. Suppose $\omega \in \Omega^s(M)$. By formulas (6.2) and (6.3) we know

$$\sum E^j \wedge i(E_l) R(E_l, E_j)\omega = \frac{1}{(s-1)!} \sum_{l,j=1}^{n} \sum_{\sigma} (-1)^{|\sigma|} \left({}^{\sigma} \left(E^j \otimes i(E_l) R(E_l, E_j)\omega \right) \right).$$

For each $\sigma \in P_s$, we associate with it the following map

$$\pi_\sigma : \{1, \cdots, \widehat{\sigma(1)}, \cdots, s\} \to \{1, \cdots, \widehat{\sigma(1)}, \cdots, s\}, \quad \sigma(n) \mapsto \begin{cases} n-1 & \text{if } 2 \leq n \leq i \\ n & \text{if } n > i \end{cases}$$

then using that $i(E_l)R(E_l,E_j)\omega$ and $R(E_l,X_i)\omega$ are differential forms, we know

$$\begin{aligned}
&\left(\sum E^j \wedge i(E_l)R(E_l,E_j)\omega\right)(X_1,\cdots,X_s)\\
=\ & \frac{1}{(s-1)!}\sum_{l,j=1}^{n}\sum_{i=1}^{s}\sum_{\substack{\sigma\in P_s\\ \sigma(1)=i}}(-1)^{|\sigma|}\left({}^{\sigma}\left(E^j\otimes i(E_l)R(E_l,E_j)\omega\right)\right)(X_1,\cdots,X_s)\\
=\ & \frac{1}{(s-1)!}\sum_{l,j=1}^{n}\sum_{i=1}^{s}\sum_{\substack{\sigma\in P_s\\ \sigma(1)=i}}(-1)^{|\sigma|}E^j(X_i)\cdot\left(i(E_l)R(E_l,E_j)\omega\right)\left(X_{\sigma(2)},\cdots,X_{\sigma(s)}\right)\\
=\ & \frac{1}{(s-1)!}\sum_{l,j=1}^{n}\sum_{i=1}^{s}\sum_{\substack{\sigma\in P_s\\ \sigma(1)=i}}(-1)^{|\sigma|}(-1)^{|\pi_\sigma|}E^j(X_i)\cdot\left(i(E_l)R(E_l,E_j)\omega\right)\left(X_1,\cdots,\widehat{X}_i,\cdots,X_s\right)\\
=\ & \frac{1}{(s-1)!}\sum_{l,j=1}^{n}\sum_{i=1}^{s}\sum_{\substack{\sigma\in P_s\\ \sigma(1)=i}}(-1)^{i-1}E^j(X_i)\cdot\left(i(E_l)R(E_l,E_j)\omega\right)\left(X_1,\cdots,\widehat{X}_i,\cdots,X_s\right)\\
=\ & \sum_{l,j=1}^{n}\sum_{i=1}^{s}(-1)^{i-1}E^j(X_i)\cdot\left(i(E_l)R(E_l,E_j)\omega\right)\left(X_1,\cdots,\widehat{X}_i,\cdots,X_s\right)\\
=\ & \sum_{l=1}^{n}\sum_{i=1}^{s}(-1)^{i-1}\left(R(E_l,X_i)\omega\right)\left(E_l,X_1,\cdots,\widehat{X}_i,\cdots,X_s\right)\\
=\ & \sum_{l=1}^{n}\sum_{i=1}^{s}\left(R(E_l,X_i)\omega\right)(X_1,\cdots,E_l,\cdots,X_s)=\mathrm{Ric}(\omega)(X_1,\cdots,X_s)
\end{aligned}$$

where we use the fact that $(-1)^{|\sigma|}(-1)^{|\pi_\sigma|}=(-1)^{i-1}$.[7] □

Remark 2.8. Clearly, we have

$$\mathrm{Ric}(f\omega)=f\,\mathrm{Ric}(\omega)\quad \forall f\in C^\infty(M). \tag{2.2}$$

In the next we simplify $\mathrm{Ric}(\omega)$ furthermore on *locally conformally flat manifolds*. This is natural, since for a conformally flat manifold M, we have

$$R=P\owedge g,\qquad \text{where } P \text{ is the Schouten tensor.}$$

which shows that the curvature tensor is easy to understand via P.

Proposition 2.9. *Let (M,g) be a locally conformally flat manifold. We regard the Schouten tensor P as a symmetric $(1,1)$-tensor $P^i{}_j$. Then for each $x\in M$, by corollary 4.3, there exists an orthogonal basis (e_i) of T_xM such that*

$$P(e_i)=\lambda_i e_i\quad \textit{for some } \lambda_i\in\mathbb{R}.$$

[7] We extend π_σ to $\widetilde{\pi}_\sigma\in P_s$ by setting $\widetilde{\pi}_\sigma(\sigma(1))=\sigma(1)$. Then $(-1)^{|\pi_\sigma|}=(-1)^{|\widetilde{\pi}_\sigma|}$, and clearly we have

$$(-1)^{|\sigma|}(-1)^{|\pi_\sigma|}=(-1)^{|\sigma|}(-1)^{|\widetilde{\pi}_\sigma|}=(-1)^{|\widetilde{\pi}_\sigma\circ\sigma|}=(-1)^{i-1}.$$

WLOG we consider $\omega = e^1 \wedge \cdots \wedge e^p$; then

$$2\mathrm{Ric}(\omega) = \left((n-p)\sum_{i=1}^{p} \lambda_i + p \sum_{i=p+1}^{n} \lambda_i \right) \omega. \tag{2.3}$$

Remark 2.10. In fact, by formula (2.2), formula (2.3) shows how to compute $\mathrm{Ric}(\omega)$ in the general sense.

Proof. As in subsection 6.B, we regard $R(e_l, e_j)$ as a derivation determined by itself as a $(1,1)$-tensor. Specifically, note that

$$\begin{aligned}
& 2R(e_l, e_j)(X, \omega) = 2(P \owedge g)\left(e_l, e_j, X, \omega^\sharp\right) \\
= \;& P(e_l, \omega^\sharp) g(e_j, X) + P(e_j, X) g(e_l, \omega^\sharp) - P(e_l, X) g(e_j, \omega^\sharp) - P(e_j, \omega^\sharp) g(e_l, X) \\
= \;& \left\langle \lambda_l e_l, \omega^\sharp \right\rangle \left\langle e_j, X \right\rangle + \left\langle \lambda e_j, X \right\rangle \left\langle e_l, \omega^\sharp \right\rangle - \left\langle \lambda_l e_l, X \right\rangle \left\langle e_j, \omega^\sharp \right\rangle - \left\langle \lambda_j e_j, \omega^\sharp \right\rangle \left\langle e_l, X \right\rangle \\
= \;& (\lambda_l + \lambda_j)\left(\left\langle e_l, \omega^\sharp \right\rangle \left\langle e_j, X \right\rangle - \left\langle e_l, X \right\rangle \left\langle e_j, \omega^\sharp \right\rangle \right)
\end{aligned}$$

and hence

$$2R(E_l, E_j) = (\lambda_l + \lambda_j)\left(e^j(\bullet) e_l - e^l(\bullet) e_j\right) = (\lambda_l + \lambda_j)\left(e^j \otimes e_l - e^l \otimes e_j\right).$$

Recall that if $e^{i_1} \wedge \cdots \wedge e^{i_s} \neq 0$ then we have

$$i(e_i)\left(e^{i_1} \wedge \cdots \wedge e^{i_s}\right) = \begin{cases} 0 & i \notin \{i_1, \cdots, i_s\} \\ (-1)^{k-1} e^{i_1} \wedge \cdots \wedge \widehat{e^{i_k}} \wedge \cdots \wedge e^{i_s} & i = i_k \end{cases}$$

then it follows from proposition 6.7 and formulas (6.4) (2.1) that

$$\begin{aligned}
2\mathrm{Ric}(\omega) &= \sum e^j \wedge i(e_l) 2R(e_l, e_j)(e^1 \wedge \cdots \wedge e^p) \\
&= \sum_{j,l=1}^{n} \sum_{k=1}^{p} e^j \wedge i(e_l)\left(e^1 \wedge \cdots \wedge \left(2R(e_l, e_j) e^k\right) \wedge \cdots \wedge e^p\right) \\
&= \sum_{j,l=1}^{n} \sum_{k=1}^{p} (\lambda_l + \lambda_j) e^j \wedge i(e_l)\left(e^1 \wedge \cdots \wedge \left(\delta_j^k e^l - \delta_l^k e^j\right) \wedge \cdots \wedge e^p\right) \\
&= \sum_{l=1}^{n} \sum_{k=1}^{p} (\lambda_l + \lambda_k) e^k \wedge i(e_l)\left(e^1 \wedge \cdots \wedge e^{k-1} \wedge e^l \wedge e^{k+1} \wedge \cdots \wedge e^p\right) \\
&\quad - \sum_{j=1}^{n} \sum_{k=1}^{p} (\lambda_k + \lambda_j) e^j \wedge i(e_k)\left(e^1 \wedge \cdots \wedge e^{k-1} \wedge e^j \wedge e^{k+1} \wedge \cdots \wedge e^p\right) \\
&= \sum_{l=p+1}^{n} \sum_{k=1}^{p} (\lambda_l + \lambda_k) e^k \wedge i(e_l)\left(e^1 \wedge \cdots \wedge e^{k-1} \wedge e^l \wedge e^{k+1} \wedge \cdots \wedge e^p\right) \\
&= \sum_{l=p+1}^{n} \sum_{k=1}^{p} (\lambda_l + \lambda_k) \omega = \left((n-p)\sum_{i=1}^{p} \lambda_i + p \sum_{i=p+1}^{n} \lambda_i \right) \omega.
\end{aligned}$$

We are done. ∎

Now, it's clear that

if the second geometric quantity $G_{n,p}(P) > 0$, then vanishing theorems follow where the second geometric quantity is introduced in subsection 5.A. In the next we use subsection 5.B, the estimates of geometric quantities, to derive vanishing theorems.

Remark 2.11. For the sake of convenience, let b_q denote the q-th Betti number, let S^{n-p} denote the standard sphere of sectional curvature 1, and let H^p denote a hyperbolic plane of sectional curvature -1.

Proposition 2.12. *Let (M,g) be a compact, locally conformally flat manifold and let $2 \le k \le n/2$ and $1 \le p \le n/2$. Suppose $g \in \overline{\Gamma}_k^+$ and $\mathrm{tr}_1(P)$ is not identical to zero in M.*
(1) If $E_{n,p} \in \Gamma_{k-1}^+$ and $E_{n,p} \notin \Gamma_k^+$, then

$$b_q = 0 \quad \textit{for} \quad p \le q \le n-p.$$

(2) Suppose $E_{n,p} \in \overline{\Gamma}_k^+$, $\sigma_k(E_{n,p}) = 0$ and $\mathrm{tr}_k(P) > 0$ at some point in M, then

$$b_q = 0 \quad \textit{for} \quad p \le q \le n-p.$$

(3) Suppose $E_{n,p} \in \overline{\Gamma}_k^+$, $\sigma_k(E_{n,p}) = 0$, then $b_p \ne 0$ iff (M,g) is a quotient of $S^{n-p} \times H^p$.

Proof. Under the conditions given in (1) and (2), the estimate of second geometric quantity (proposition 5.10) implies that

$$\mathrm{Ric} : \Omega^p(M) \to \Omega^p(M)$$

is a non-negative operator and positive at some point. Therefore, by Bochner technique (lemma 6.19) and Hodge theorem 6.23, we know $b_q = 0$ for $p \le q \le n-p$.

In the next we prove point (3). By Hodge theorem 6.23, there exists a non-zero harmonic p-from ω. Again, the estimate of second geometric quantity (proposition 5.10) implies that

$$\mathrm{Ric} : \Omega^p(M) \to \Omega^p(M)$$

is a non-negative operator. Therefore, by Bochner technique (lemma 6.19), ω is parallel.

After showing the existence of such parallel and non-zero harmonic p-from ω, [GLW05] claims that the conclusion follows form a technique of holonomy group. □

Theorem 2.13. *Let (M,g) be a compact, locally conformally flat manifold with $\mathrm{tr}_1(P) > 0$.*
(1) If $g \in \overline{\Gamma}_k^+$ for some $2 \le k < n/2$, then

$$b_q = 0 \quad \textit{for} \quad \left[\frac{n+1}{2}\right] + 1 - k \le q \le n - \left(\left[\frac{n+1}{2}\right] + 1 - k\right).$$

(2) Suppose $g \in \Gamma_2^+$, then

$$b_q = 0 \quad \textit{for} \quad \left[\frac{n-\sqrt{n}}{2}\right] \le q \le \left[\frac{n+\sqrt{n}}{2}\right].$$

If $g \in \overline{\Gamma}_2^+$ and $b_p \ne 0$ where $p = \frac{n-\sqrt{n}}{2}$, then (M,g) is a quotient of $S^{n-p} \times H^p$.
(3) If $g \in \Gamma_k^+$ for some $k \ge \frac{n-\sqrt{n}}{2}$, then

$$b_q = 0 \quad \textit{for} \quad 2 \le q \le n-2.$$

If $g \in \overline{\Gamma}_k^+$ and $b_2 \ne 0$ where $k = \frac{n-\sqrt{n}}{2}$, then (M,g) is a quotient of $S^{n-2} \times H^2$.

Proof. It follows from proposition 2.12 and proposition 5.11. □

3. CONFORMAL DEFORMATION OF SCALAR CURVATURES

3.A. Introduction.

The core problem is as follows:

Problem 3.1. *Let (M, g) be a closed Riemannian manifold with dimension $n \geq 2$, and let*

$$\mathcal{C}_g = \{\rho g : \rho \in C^\infty(M),\ \rho > 0\}.$$

Given $h \in C^\infty(M)$. Does there exist $\widetilde{g} \in \mathcal{C}_g$ such that $\widetilde{\text{scal}} = h$?

By formula (1.10), we know

$$\widetilde{\text{scal}} = e^{-2f}\left(\text{scal} - 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2\right)$$

for $\widetilde{g} = e^{2f} g$. Therefore,

(1) If $n = 2$, for $\widetilde{g} = e^{2u} g$ we have

$$\widetilde{\text{scal}} = e^{-2u}\left(\text{scal} - 2\Delta u\right). \tag{3.1}$$

(2) If $n \geq 3$, for $\widetilde{g} = u^{\frac{4}{n-2}} g$ we have

$$\widetilde{\text{scal}} = u^{-\frac{n+2}{n-2}}\left(\text{scal} \cdot u - \frac{4(n-1)}{n-2}\Delta u\right). \tag{3.2}$$

Remark 3.2. For the case that $n \geq 3$, we set $f = \phi(u)$ to kill the gradient term, where ϕ is to be determined. Namely, since

$$\Delta\phi(u) = \dot{\phi}(u) + \ddot{\phi}(u)|\nabla u|^2 \quad \text{and} \quad |\nabla\phi(u)|^2 = \dot{\phi}(u)^2|\nabla u|^2,$$

we need to find ϕ with

$$2\ddot{\phi} + (n-2)\dot{\phi}^2 = 0.$$

Then we set $\phi = \frac{2}{n-2}\log u$.

If h is constant, then problem 3.1 becomes the Yamabe problem. In dimension $n = 2$, Yamabe problem follows from the uniformization theorem. In dimension $n \geq 3$, Yamabe problem is solved by Yamabe (1960), Trüdinger (1968), Aubin (1976), and Schoen (1984).

Theorem 3.3 (Yamabe, Trüdinger, Aubin)**.** *The Yamabe problem can be solved on any closed manifold M with $\lambda(M) < \lambda(S^n)$, where S^n is the sphere with its standard metric and*

$$\lambda(M) = \inf_{\widetilde{g} \in \mathcal{C}_g} \mathcal{Q}(\widetilde{g}), \quad \textit{where} \quad \mathcal{Q}(\widetilde{g}) = \frac{\int_M \widetilde{\text{scal}}\, \text{dvol}_{\widetilde{g}}}{\left(\int_M \text{dvol}_{\widetilde{g}}\right)^{1-\frac{2}{n}}}.$$

Theorem 3.4 (Aubin)**.** *If M has dimension $n > 6$ and is not locally conformally flat then $\lambda(M) < \lambda(S^n)$.*

Theorem 3.5 (Schoen)**.** *If M has dimension 3, 4, or 5, or if M is locally conformally flat, then $\lambda(M) < \lambda(S^n)$ unless M is conformal to the standard sphere.*

Yamabe put forward his problem in order to prove the Poincaré conjecture. By corollary 1.16, to prove the Poincaré conjecture, it suffices to show that any simply connected 3-dimension manifold admits an Einstein metric. Clearly, Yamabe problem is our first step.

Moreover, let $\mathcal{M}$ be the colletion of all Riemannian metrics on M, and we set

$$\lambda(M,g) = \inf_{\widetilde{g}\in\mathcal{C}_g} \mathcal{Q}(\widetilde{g}), \quad \text{and} \quad \Lambda(M) = \sup_{g\in\mathcal{M}} \inf_{\widetilde{g}\in\mathcal{C}_g} \mathcal{Q}(\widetilde{g}).$$

Then we have:

(1) If $\widetilde{g}\in\mathcal{C}_g$ with $\mathcal{Q}(\widetilde{g}) = \lambda(M,g)$, then $\widetilde{g}$ has constant scalar curvature; [Euler-Lagrange equation]
(2) If $g\in\mathcal{M}$ with $\mathcal{Q}(g) = \lambda(M,g) = \Lambda(M)$, then g is Einstein.

We say that g achieves $\Lambda(M)$ if $\mathcal{Q}(g) = \lambda(M,g) = \Lambda(M)$. Then the standard metric g_1 on S^n with $n\geq 3$ achieves $\Lambda(S^n)$, and the standard metric g_0 on T^n achieves $\Lambda(S^n)$.[8]

It's still unknown that whether the Poincaré metric g_{-1} achieves $\Lambda(\mathbb{H}^n)$.

3.B. **The two dimensional cases.** First we consider the case that $\widetilde{\mathrm{scal}}$ is constant (in dimension 2). As we said before, we can solve (3.1) by the uniformization theorem.

Theorem 3.6 (Uniformization theorem)**.** *Every simply connected Riemann surface is biholomorphic to one of three Riemann surfaces: the open unit disk, the complex plane, or the Riemann sphere.*

Proof. One can refer to [Cha]. □

Corollary 3.7. *Let M be an orientable closed 2-dimensional Riemannian manifold. Then M admits a conformally equivalent metric of constant curvature.*

Proof. It's well-known that a Riemann surface with a complex structure corresponds to a 2-dimensional oriented manifold with orientation-preserving isothermal coordinate charts, and that biholomorphic maps correspond to conformal transformations.

By this correspondence and theorem 3.6, each such is conformally equivalent to a unique closed 2-manifold of constant curvature, so a quotient of one of the following by a free action of a discrete subgroup of an isometry group:

(1) the sphere (curvature $+1$);
(2) the Euclidean plane (curvature 0);
(3) the hyperbolic plane (curvature -1).

Hence we get the conclusion by the classification of closed orientable Riemannian 2-manifolds. □

In the next we consider problem 3.1 for general $\widetilde{\mathrm{scal}}$ (in dimension 2). Our equation (3.1) becomes

$$\Delta u - K + \widetilde{K}e^{2u} = 0 \tag{3.3}$$

where K is the Gaussian curvature, and $\widetilde{K}$ is a given function.

First we note that Gauss-Bonnet formula yields that

$$\int_M K\,\mathrm{dvol}_g = 2\pi\chi(M).$$

[8] There are no metrics on T^n with positive scalar curvature. See [Li] and [GL83].

If u solves (3.3), then we have

$$\int_M \widetilde{K} e^{2u} \, \mathrm{dvol}_g = 2\pi\chi(M). \tag{3.4}$$

This is just the Gauss-Bonnet formula for $(M, \widetilde{g})$, since $\mathrm{dvol}_{\widetilde{g}} = e^{2u}\,\mathrm{dvol}_g$[9] and $\widetilde{K}$ is exactly the Gaussian curvature for $\widetilde{g}$.

Clearly, in cases where $\chi(M)$ has different signs the given function $\widetilde{K}$ should satisfy different kinds of conditions. Hence, we separate our discussions into three cases according to whether $\chi(M)$ is negative, zero, or positive.

Case 1: $\chi(M) < 0$. In this case, although the existence problem of (3.3) has not been completely resolved, we have a relatively good understanding for the problem. For this case it seems reasonable to solve (3.3) by the so-called principle of "sup- and sub-solutions". The following is a simple case of this principle.

Proposition 3.8. *Let (M, g) be a smooth, compact, Riemannian manifold. Consider the semi-linear elliptic equation*

$$\Delta u + f(x, u) = 0 \tag{3.5}$$

where $f \in C^\infty(M \times \mathbb{R})$. Suppose that there exist $\phi, \psi \in C^2(M)$ satisfying

$$\begin{aligned} \Delta\phi + f(x, \phi) &\geq 0, \\ \Delta\psi + f(x, \psi) &\leq 0, \end{aligned} \tag{3.6}$$

(such ϕ and ψ are called respectively a sub-solution and a sup-solution for (3.5)), and $\phi \leq \psi$. Then (3.5) has a solution $u \in C^\infty(M)$ such that $\phi \leq u \leq \psi$.

Proof. The idea is as follows: we use the linearized operator to derive an approximation sequence, and then use elliptic theory to show the regularity.

Find a constant A with $-A \leq \phi \leq \psi \leq A$, and find a sufficiently large c such that

$$F(x, t) = ct + f(x, t) \quad \text{is increasing in } t \in [-A, A] \text{ for any fixed } x \in M. \tag{3.7}$$

Since $c \geq 0$, the linearized elliptic operator (where $0 < \alpha < 1$)

$$Lu = -\Delta u + cu \,:\, C^{2,\alpha}(M) \to C^{0,\alpha}(M)$$

is invertible (see theorem 10.16). Moreover, by the maximum principle, L is a positive operator, i.e.

$$Lv_1 \geq Lv_2 \implies v_1 \geq v_2. \tag{3.8}$$

Now we define inductivey

$$\begin{aligned} \phi_0 = \phi, &\quad \phi_k = L^{-1}(F(x, \phi_{k-1})), \quad k \geq 1; \\ \psi_0 = \psi, &\quad \psi_k = L^{-1}(F(x, \psi_{k-1})), \quad k \geq 1. \end{aligned} \tag{3.9}$$

Then

$$(3.6)(3.7)(3.9) \implies L\phi \leq L\phi_1 = F(x, \phi) \leq F(x, \psi) = L\psi_1 \leq L\psi$$

[9]This claims easily follows from $\mathrm{vol} = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n$.

and hence by positivity (3.8) we know

$$\phi \le \phi_1 \le \psi_1 \le \psi.$$

Similarly, by induction we know

$$\phi \le \phi_{k-1} \le \phi_k \le \psi_k \le \psi_{k-1} \le \psi, \quad \forall k \ge 1.$$

Then pointwisely, $\phi_k \to \underline{u}$ and $\psi_k \to \overline{u}$ with $\phi \le \underline{u} \le \overline{u} \le \psi$.

(1) By lemma 8.9, the pointwise convergence is in fact a convergence in $L^p(M)$ for all $1 \le p < \infty$;
(2) By L^p estimate 10.15, the convergence in $L^p(M)$ is in fact a convergence in $L^p_2(M)$.
(3) Taking a sufficiently large p, then Sobolev embedding theorem 10.13 implies that the convergence in $L^p_2(M)$ is in fact a convergence in $C^{0,\alpha}(M)$.
(4) By formula (3.9) and Schauder estimate 10.16, the convergence in $C^{0,\alpha}(M)$ is in fact a convergence in $C^\infty(M)$.

Therefore, taking limit we get

$$Lv = F(x, v)$$

where $v = \underline{u}$ or $\overline{u}$ and $v \in C^\infty(M)$. Then we get the conclusion. □

Now we come back to our equation (3.3).

Proposition 3.9. *Suppose $\chi(M) < 0$. Then a sufficient condition for the existence of a solution of (3.3) is that there exists a sup-solution $\psi \in C^2(M)$ for (3.3).*

Proof. By proposition 3.8, it suffices to find a sub-solution ϕ for (3.3) such that $\phi \le \psi$. Note that

$$K_0 := \frac{\int_M K \,\mathrm{dvol}}{\int_M \mathrm{dvol}} \implies \int_M (K - K_0)\,\mathrm{dvol} = 0.$$

By corollary 6.24, $K - K_0 = \Delta f$ for some $f \in C^\infty(M)$. Setting $\phi = f - c$ for sufficiently large c, then $\phi \le \psi$. Note that

$$\Delta\phi - K + \widetilde{K}e^{2\phi} = -K_0 + \widetilde{K}e^{2f-2c}$$

and that

$$\chi(M) < 0 \implies K_0 < 0 \quad \text{(Gauss-Bonnet)}.$$

Therefore, pick a sufficiently large c, we get a sub-solution as desired. We are done. □

Corollary 3.10 (Kazdan-Warner). *If $\chi(M) < 0$, $\widetilde{K} \le 0$ but $\widetilde{K}$ is not identically zero. Then (3.3) has a solution $u \in C^\infty(M)$.*

Proof. By proposition 3.9, it suffices to find a sup-solution. By corollary 6.24, there exists $f \in C^\infty(M)$ solves

$$\Delta f = \widetilde{K}_0 - \widetilde{K}$$

where $\widetilde{K}_0$ is the mean value of $\widetilde{K}$. We set $\psi = af + b$ where a and b are to be determined. By condition, $\widetilde{K}_0 < 0$. Pick sufficiently large a and b such that

$$a\widetilde{K}_0 < K(x), \forall x \in M \quad \text{and} \quad e^{af+b} - a > 0.$$

Then

$$\Delta\psi - K + \widetilde{K}e^{\psi} = a\widetilde{K}_0 - K + \left(e^{af+b} - a\right)\widetilde{K} < 0,$$

which shows that ψ is a sup-solution. We are done. □

Note that condition (3.4) indicates that, in the case $\chi(M) < 0$, for (3.3) to have a solution it is necessary that $\widetilde{K}$ takes negative values somewhere. However, if $\widetilde{K}$ changes sign it may happen that (3.3) has no solutions. In such a case, we do not know the necessary and sufficient conditions for the solvability of (3.3).

Case 2: $\chi(M) = 0$. This case has been completely solved.

Theorem 3.11. *Assume $\chi(M) = 0$. Then (3.3) has a smooth solution iff either*
(1) $\widetilde{K} \equiv 0$, or
(2) $\widetilde{K}$ changes sign and satisfies

$$\int_M \widetilde{K}e^{2f}\,\mathrm{dvol} < 0, \tag{3.10}$$

where f is a solution to $\Delta f = K$.

Proof. Necessity. Note that

$$\begin{aligned} \chi(M) = 0 \quad &\Longrightarrow \quad \int_M K\,\mathrm{dvol} = 0 \quad \text{(Gauss-Bonnet)}, \\ &\Longrightarrow \quad \exists f \in C^\infty(M) \quad \text{with} \quad K = \Delta f \quad \text{(corollary 6.24)}. \end{aligned}$$

If u is a solution to (3.3), then setting $v = u - f$, we get

$$\Delta v + \widetilde{K}e^{2v+2f} = 0. \tag{3.11}$$

Therefore,

$$\begin{aligned} \int_M \widetilde{K}e^{2f}\,\mathrm{dvol} &= -\int_M e^{-2v}\Delta v\,\mathrm{dvol} = \int_M \left[\langle\nabla(e^{-2v}), \nabla v\rangle - \mathrm{div}(e^{-2v}\nabla v)\right]\mathrm{dvol} \\ &= -2\int_M e^{-2v}|\nabla v|^2\,\mathrm{dvol} \le 0. \end{aligned}$$

If this integration equals to zero, clearly v is constant, and hence (3.11) implies that $\widetilde{K} \equiv 0$. Therefore, if $\widetilde{K} \not\equiv 0$, then (3.10) holds, and clearly $\widetilde{K}$ changes sign.

Sufficiency. If $\widetilde{K} \equiv 0$, then f is the solution. In the next we assume that $\widetilde{K} \not\equiv 0$.

It suffices to find $v = u - f$ that satisfies (3.11). The idea is to apply the method of Lagrange multiplier 8.2 and the variational method, which transfers the equation to a minimizer problem.

Specifically, set

$$\mathcal{A} = \left\{\phi \in L_1^2(M) : \int_M \phi\,\mathrm{dvol} = \int_M \widetilde{K}e^{2\phi+2f}\,\mathrm{dvol} = 0\right\}$$

and

$$J : L_1^2(M) \to \mathbb{R}, \quad \phi \mapsto \frac{1}{2}\int_M |\nabla\phi|^2.$$

We consider the problem $\inf_{\mathcal{A}} J$.

(1) First we prove that there exists $v \in \mathcal{A}$ such that $J(v) = \inf_{\mathcal{A}} J$.
Suppose (ϕ_i) is a sequence in $\mathcal{A}$ such that

$$J(\phi_i) \to \inf_{\mathcal{A}} J.$$

Clearly, Poincaré inequality 10.23 implies that (ϕ_i) is bounded in $L_1^2(M)$. Since $L_1^2(M)$ is reflexive, by [Xio, theorem 3.41], there exists a subsequence, which we relabel as (ϕ_i), satisfying

$$\phi_k \rightharpoonup v \quad \text{in} \quad L_1^2(M)$$

for some $v \in L_1^2(M)$. Since J is weakly lower semi-continuous (see remark 10.22),

$$J(v) \leq \liminf_{i \to \infty} J(\phi_i) = \inf_{\mathcal{A}} J.$$

The Sobolev embedding theorem 10.13 implies that $\phi_k \to v$ in $L^p(M)$ for any $p \geq 1$, and hence

$$\int_M v \, \mathrm{dvol} = \lim_{i \to \infty} \int_M \phi_i \, \mathrm{dvol} = 0.$$

Moreover, by the subsequent technique lemma (corollary 3.15), we know

$$\int_M \widetilde{K} e^{2v+2f} \, \mathrm{dvol} = \lim_{i \to \infty} \int_M \widetilde{K} e^{2\phi_i + 2f} \, \mathrm{dvol} = 0.$$

Therefore $v \in \mathcal{A}$. Hence we find a minimizer.

(2) Then we show that the minimizer v (up to a constant difference) is a solution to (3.11) in the sense of $L_1^2(M)$-weak solution.
Applying the method of Lagrange multipliers 8.2,[10] there exists $\alpha, \beta \in \mathbb{R}$ such that v is a critical point of

$$\widetilde{J} : L_1^2(M) \to \mathbb{R}, \quad \phi \mapsto \frac{1}{2} \int_M |\nabla \phi|^2 \, \mathrm{dvol} - \alpha \int_M \phi \, \mathrm{dvol} - \beta \int_M \widetilde{K} e^{2\phi+2f} \, \mathrm{dvol}.$$

Note that for any $\phi \in L_1^2(M)$, we have

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \widetilde{J}(v + t\phi) \\ &= \int_M \left(\langle \nabla v, \nabla \phi \rangle - \alpha \phi - 2\beta \widetilde{K} e^{2v+2f} \phi \right) \mathrm{dvol} \\ &= -\int_M \left(\Delta v + \alpha + 2\beta \widetilde{K} e^{2v+2f} \right) \phi \, \mathrm{dvol} \end{aligned}$$

Therefore, in the sense of $L_1^2(M)$-weak solution, v satisfies

$$\Delta v + \alpha + 2\beta \widetilde{K} e^{2v+2f} = 0.$$

[10] Set $f = J : L_1^2(M) \to \mathbb{R}$ and define $g : L_1^2(M) \to \mathbb{R}^2$ by

$$g(\phi) = \left(\int_M \phi \, \mathrm{dvol}, \int_M \widetilde{K} e^{2\phi+2f} \, \mathrm{dvol} \right).$$

Then apply the method of Lagrange multipliers 8.2 to f and g.

Taking integration we know $\alpha = 0$ (since $v \in \mathcal{A}$). Also note that[11]

$$\begin{aligned}\beta \int_M \widetilde{K} e^{2f}\, \mathrm{dvol} &= -\frac{1}{2}\int_M e^{-2v}\Delta v\, \mathrm{dvol} = \frac{1}{2}\int_M \langle \nabla(e^{-2v}), \nabla v\rangle - \mathrm{div}(e^{-2v}\nabla v)\, \mathrm{dvol} \\ &= -\int_M e^{-2v}|\nabla v|^2\, \mathrm{dvol} < 0\end{aligned}$$

where the last inequality is strict since if v is contant then clear $\widetilde{K} \equiv 0$. Then the condition 3.10 implies $\beta > 0$. Therefore, setting

$$v_0 = v + \frac{1}{2}\log\beta$$

then v_0 is a solution to (3.11) in the sense of $L^2_1(M)$-weak solution.

(3) Finally we show that the solution is smooth.
By the subsequent lemma 3.14, since $v_0 \in L^2_1(M)$, we have $e^{v_0} \in L^p(M)$ for any $p \geq 1$. Then remark 10.17 (or theorem 9.24) implies that $v_0 \in C^\infty$. Then $u = v_0 + f$ is the desired smooth solution.

We are done. □

Lemma 3.12 (Trüdinger inequality). *Let (M, g) be a closed 2-dimensional Riemannian manifold. Then there exist positive constants β and C such that*

$$\int_M e^{\beta u^2}\, \mathrm{dvol} \leq C \quad \forall u \in \left\{u \in L^2_1(M) : \int_M u\, \mathrm{dvol} = 0, \int_M |\nabla u|^2\, \mathrm{dvol} \leq 1\right\}.$$

Proof. Let $(\rho_i)_{i=1}^k$ be a partition of unity subordinate to an open cover $(U_i)_{i=1}^k$ of M, where each (U_i, ϕ_i) is a unit coordinate ball, i.e. each $\phi_i(U_i)$ is the unit disk in $\mathbb{R}^2$. The idea is to show the generalization of Poincaré inequality:

$$\|u\|_{L^p(M)} \lesssim \sqrt{p}\|\nabla u\|_{L^2(M)}, \quad \forall p \geq 2 \tag{3.12}$$

and then the conclusion will follow from Taylor expansion.

(1) First we prove the generalization of Poincaré inequality

$$\|v\|_{L^p(D)} \lesssim \sqrt{p}\|\nabla_0 v\|_{L^2(D)} \quad \forall v \in W^{1,2}_0(D) \quad \forall p \geq 2$$

on the unit open disk $D \subset \mathbb{R}^2$.[12]
For each $x \in D$ we set

$$\phi_x : \mathbb{R}^2 \to \mathbb{R}, \quad y \mapsto \frac{1}{2\pi}\log(|x - y|).$$

It's well-known that

$$\Delta\phi_x = \delta(x) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^2)$$

and hence

$$v(x) = \int_D (v\Delta_0\phi_x)(y)\, dy \quad \forall v \in C^1_0(D).$$

[11] The following integration may seem a little unreasonable, but we can apply the method of step (3) to show that v is in fact a smooth solution, and then this integration becomes natural.
[12] In this point, let ∇_0 denote the standard gradient on $\mathbb{R}^2$, let Δ_0 denote the standard Laplacian on $\mathbb{R}^2$, and let div_0 denote the standard divergence on $\mathbb{R}^2$.

Therefore, via integration by parts we know for any $v \in C_0^1(D)$ we have

$$\begin{aligned} v(x) &= \frac{1}{2\pi}\int_D (\nabla_0 v)(y)\cdot \frac{x-y}{|x-y|^2}\,dy \\ &\le \frac{1}{2\pi}\int_D |\nabla_0 v(y)|\cdot|x-y|^{-1}\,dy. \end{aligned}$$

The idea is to apply the Hölder inequality to kill the term $|x-y|^{-1}$, and the key point is that we need to eliminate the term $|x-y|^{-1}$ gradually through two integrations.[13] Namely, setting

$$A(y) = |\nabla_0 v(y)|^2 \quad \text{and} \quad B(x,y) = |x-y|^{-q} \quad \text{where} \quad q = \frac{2p}{p+2} \in [1,2)$$

we know that

$$\int_D B(x,y)\,dy \le \int_{|y|<2} |y|^{-q}\,dy = 2\pi\int_0^2 r^{1-q}\,dr = 2^{1-q}\pi(p+2) \le C_1 p$$

and that

$$|\nabla_0 v(y)|\cdot|x-y|^{-1} = (A(y)B(x,y))^{1/p}\cdot B(x,y)^{1/2}\cdot A(y)^{1/2-1/p}.$$

Therefore, for any $v \in C_0^1(D)$ and $p \ge 2$ we have

$$\begin{aligned} |v(x)| &\le \frac{1}{2\pi}\left(\int_D A(y)B(x,y)\,dy\right)^{1/p}\left(\int_D B(x,y)\,dy\right)^{1/2}\left(\int_D A(y)\,dy\right)^{1/2-1/p} \\ &\le C_2\sqrt{p}\left(\|\nabla_0 v\|_{L^2(D)}\right)^{1-2/p}\cdot\left(\int_D A(y)B(x,y)\,dy\right)^{1/p} \end{aligned}$$

and hence for any $v \in C_0^1(D)$ and $p \ge 2$ we have

$$\begin{aligned} \|v\|_{L^p(D)} &\le C_2\sqrt{p}\left(\|\nabla_0 v\|_{L^2(D)}\right)^{1-2/p}\cdot\left(\int_D\int_D A(y)B(x,y)\,dy\,dx\right)^{1/p} \\ &= C_2\sqrt{p}\left(\|\nabla_0 v\|_{L^2(D)}\right)^{1-2/p}\cdot\left(\int_D A(y)\,dy\int_D B(x,y)\,dx\right)^{1/p} \\ &\le C_2\sqrt{p}\left(\|\nabla_0 v\|_{L^2(D)}\right)^{1-2/p}\cdot\left(C_1 p\int_D A(y)\,dy\right)^{1/p} \\ &\le C_3\sqrt{p}\,\|\nabla_0 v\|_{L^2(D)}. \end{aligned}$$

Since $C_0^1(D)$ is dense in $W_0^{1,2}(D)$, we know

$$\|v\|_{L^p(D)} \le C_3\sqrt{p}\|\nabla_0 v\|_{L^2(D)} \quad \forall v \in W_0^{1,2}(D) \quad \forall p \ge 2$$

for some constant C_3.

(2) Then we show the generalization of Poincaré inequality

$$\|u\|_{L^p(M)} \preceq \sqrt{p}\|\nabla u\|_{L^2(M)} \quad \forall u \in L_1^2(M) \quad \forall p \ge 2$$

[13] Since $\int_D |x-y|^{-2}\,dy$ is not under controll, we can't eliminate the term $|x-y|^{-1}$ directly by just one integration via Hölder inequality.

on a closed manifold M.
Setting $u_i = \rho_i u$, then by compactness and point (1) there exists constant C_4 with

$$\|u_i\|_{L^p(M)} \le C_4\sqrt{p}\,\|\nabla u_i\|_{L^2(M)} \quad \forall 1 \le i \le k \quad \forall p \ge 2.$$

It follows that

$$\|u\|_{L^p(M)} \le \sum_{i=1}^{k} \|u_i\|_{L^p(M)} \le C_4\sqrt{p}\sum_{i=1}^{k} \|\nabla u_i\|_{L^2(M)}$$

Note that

$$\|\nabla u_i\|_{L^2(M)} = \|\nabla(\rho_i u)\|_{L^2(M)} = \|u \cdot \nabla\rho_i\|_{L^2(M)} + \|\rho_i \cdot \nabla u\|_{L^2(M)}.$$

Then by finiteness and compactness there exists constant C_5 with

$$\|\nabla u_i\|_{L^2(M)} \le C_5\,\|u\|_{L^2(M)} + \|\nabla u\|_{L^2(M)} \quad \forall 1 \le i \le k$$

and hence by the standard Poincaré inequality 10.23 we know (since $\int_M u\,\mathrm{dvol} = 0$)

$$\|u\|_{L^p(M)} \le C_6\sqrt{p}\,\|\nabla u\|_{L^2(M)}.$$

(3) Finally we prove the conclusion via Taylor expansion.
Now it follows from $\|\nabla u\|_{L^2(M)} \le 1$ that

$$\int_M \sum_{k=0}^{N} \frac{1}{k!}\left(\beta|u|^2\right)^k \mathrm{dvol} = \sum_{k=0}^{N} \frac{\beta^k}{k!}\int |u|^{2k}\,\mathrm{dvol} \le \sum_{k=0}^{N} \frac{k^k}{k!}\left(2C_6^2\beta\right)^k$$

Since Stirling's approximation yields that

$$\frac{k^k}{k!}\left(2C_6^2\beta\right)^k = \frac{\left(2C_6^2 e\beta\right)^k}{\sqrt{2\pi k}}\left(1 + O\left(\frac{1}{k}\right)\right).$$

we know

$$\beta < \frac{1}{2C_6^2 e} \implies \sum_{k=0}^{\infty} \frac{k^k}{k!}\left(2C_6^2\beta\right)^k < \infty.$$

Then the conclusion follows from Taylor expansion and the monotone convergence theorem [Xio, theorem 9.10].

We are done. □

Remark 3.13. For the best constant in Trüdinger inequality, one can refer to [Mos71].

Lemma 3.14. *Let (M, g) be a closed 2-dimensional manifold. There exist constants $C, \eta > 0$ such that*

$$\int_M e^u\,\mathrm{dvol} \le C\exp\left(\eta\|\nabla u\|_{L^2(M)}^2 + \frac{1}{V}\int_M u\,\mathrm{dvol}\right) \tag{3.13}$$

where $V = \int_M \mathrm{dvol}$ is the volume of M. Moreover, $e^u \in L^p(M)$ for any $p \ge 1$.

Proof. WLOG we assume $\|\nabla u\|_{L^2(M)} \ne 0$; otherwise u is constant and the conclusion is trivial. Setting

$$u_0 = u - \frac{1}{V}\int_M u\,\mathrm{dvol} \quad \text{and} \quad \phi = \frac{u_0}{\|\nabla u\|_{L^2(M)}}$$

then by Trüdinger inequality 3.12, there exist positive constants β and C such that

$$\int e^{\beta\phi^2}\,\mathrm{dvol} \le C. \tag{3.14}$$

Since

$$u_0 \le \beta\left(\frac{u_0}{\|\nabla u\|_{L^2(M)}}\right)^2 + \|\nabla u\|^2_{L^2(M)}/4\beta$$

we have

$$\begin{aligned}\int e^u\,\mathrm{dvol} &= \exp\left(\frac{1}{V}\int_M u\,\mathrm{dvol}\right)\cdot\int e^{u_0}\,\mathrm{dvol}\\ &\le \exp\left(\frac{1}{V}\int_M u\,\mathrm{dvol} + \|\nabla u\|^2_{L^2(M)}/4\beta\right)\cdot\int e^{\beta\phi^2}\,\mathrm{dvol}\end{aligned}$$

which implies (3.13) together with Trüdinger inequality (3.14). The last assertion follows by replacing u with pu. □

Corollary 3.15. *Let (M,g) be a closed 2-dimensional manifold. Consider the map*

$$I : L^2_1(M) \to \mathbb{R}, \quad u \mapsto \int_M f e^u\,\mathrm{dvol}$$

where $f \in C^\infty(M)$. Then I is continuous with respect to the weak topology of $L^2_1(M)$.

Proof. Suppose $u_i \rightharpoonup u$ in $L^2_1(M)$. Then by Sobolev embedding theorem 10.13, $u_i \to u$ in $L^p(M)$ for any $p \ge 1$. Therefore, by Fubini theorem [Xio, theorem 9.28] we know

$$\begin{aligned}\int_M f(e^{u_i} - e^u)\,\mathrm{dvol} &= \int_M\int_0^1 f e^{u+t(u_i-u)}(u_i-u)\,dt\,\mathrm{dvol}\\ &= \int_0^1\int_M f e^{u+t(u_i-u)}(u_i-u)\,\mathrm{dvol}\,dt.\end{aligned}$$

Then by Hölder inequality and lemma 3.14, we know $I(u_i) \to I(u)$. We are done. □

Case 3: $\chi(M) > 0$. In this case, $M = S^2$ or $\mathbb{RP}^2$. First we consider the case $(M,g) = (S^2, g_0)$ where g_0 is the standard metric. Then it has Gaussian curvature $K \equiv 1$. Moreover, equation (3.3) becomes

$$\Delta u - 1 + \widetilde{K}e^{2u} = 0 \tag{3.15}$$

and condition (3.4) becomes

$$\int_{S^2}\widetilde{K}e^{2u}\,\mathrm{dvol} = 4\pi. \tag{3.16}$$

This requires that $\widetilde{K}$ must take positive values somewhere. However, even if $\widetilde{K} > 0$, (3.15) sitll may have no solutions.

Proposition 3.16 (Kazdan-Warner). *Let ϕ be a first eigenfunction on standard sphere:*

$$\Delta\phi + 2\phi = 0. \tag{3.17}$$

Suppose $u \in C^\infty(S^2)$ is a solution to (3.15). Then

$$\int_{S^2} \langle \nabla \widetilde{K}, \nabla \phi \rangle e^{2u} \,\mathrm{dvol} = 0 \tag{3.18}$$

Remark 3.17. By [Li12, theorem 5.1], the first (positive) eigenvalue of S^n is n.

Remark 3.18. Considering $\widetilde{K} = 1 + \varepsilon\phi$ for sufficiently small ε, then $\widetilde{K} > 0$. But (3.18) indicates that (3.15) doesn't have a solution for this $\widetilde{K}$.

Proof. Let (E_i) be any local orthonormal frame. By the proof of [Li12, theorem 5.1], we know that ϕ satisfies

$$\phi_{;i;j} = -\phi\delta_{ij}. \tag{3.19}$$

Multiplying equation (3.15) by $\langle \nabla u, \nabla \phi \rangle$ and then integrating over S^2 we get

$$\int_{S^2} \langle \nabla u, \nabla \phi \rangle \Delta u \,\mathrm{dvol} - \int_{S^2} \langle \nabla u, \nabla \phi \rangle \,\mathrm{dvol} + \int_{S^2} \langle \nabla u, \nabla \phi \rangle \widetilde{K} e^{2u} \,\mathrm{dvol} = 0.$$

It follows from (3.19) that

$$\begin{aligned}
&\int_{S^2} \langle \nabla u, \nabla \phi \rangle \Delta u \,\mathrm{dvol} = -\int_{S^2} \langle \nabla \langle \nabla u, \nabla \phi \rangle, \nabla u \rangle \,\mathrm{dvol} \\
= &-\int_{S^2} \left(u_{;1;1}u_{;1}\phi_{;1} - u_{;1}u_{;1}\phi + u_{;2;1}u_{;1}\phi_{;2} + u_{;1;2}u_{;2}\phi_{;1} - u_{;2}u_{;2}\phi + u_{;2;2}u_{;1}\phi_{;2}\right) \mathrm{dvol} \\
= &-\frac{1}{2}\int_{S^2} \left\langle \nabla \left(|\nabla u|^2\right), \nabla \phi \right\rangle \mathrm{dvol} + \int_{S^2} |\nabla u|^2 \phi \,\mathrm{dvol} \\
= &\ \frac{1}{2}\int_{S^2} |\nabla u|^2 (\Delta\phi + 2\phi) \,\mathrm{dvol} = 0.
\end{aligned}$$

Note that

$$\int_{S^2} \langle \nabla u, \nabla \phi \rangle \,\mathrm{dvol} = -\int_{S^2} \phi \Delta u \,\mathrm{dvol} = \int_{S^2} \phi \left(\widetilde{K} e^{2u} - 1\right) \mathrm{dvol} = \int_{S^2} \phi \widetilde{K} e^{2u} \,\mathrm{dvol}$$

and that

$$\begin{aligned}
\int_{S^2} \langle \nabla u, \nabla \phi \rangle \widetilde{K} e^{2u} \,\mathrm{dvol} &= \frac{1}{2}\int_{S^2} \left\langle \widetilde{K} \nabla e^{2u}, \nabla \phi \right\rangle \mathrm{dvol} = \frac{1}{2}\int_{S^2} \left\langle \nabla \left(\widetilde{K} e^{2u}\right) - e^{2u} \nabla \widetilde{K}, \nabla \phi \right\rangle \mathrm{dvol} \\
&= -\frac{1}{2}\int_{S^2} \widetilde{K} e^{2u} \Delta \phi \,\mathrm{dvol} - \frac{1}{2}\int_M \left\langle \nabla \widetilde{K}, \nabla \phi \right\rangle e^{2u} \,\mathrm{dvol} \\
&= \int_{S^2} \phi \widetilde{K} e^{2u} \,\mathrm{dvol} - \frac{1}{2}\int_M \left\langle \nabla \widetilde{K}, \nabla \phi \right\rangle e^{2u} \,\mathrm{dvol}.
\end{aligned}$$

Then the conclusion follows. □

Remark 3.19. Generally, on (S^n, g_0) we have similar conclusions like formula (3.18). Namely, via a similar process one can prove that

$$\int_{S^2} \left\langle \nabla \phi, \nabla \widetilde{\mathrm{scal}} \right\rangle u^{\frac{2n}{n-2}} \,\mathrm{dvol} = 0$$

where ϕ is the first eigenfunction on S^n (i.e. $\Delta\phi + n\phi = 0$) and $\widetilde{g} = u^{\frac{4}{n-2}} g_0$.

Moreover, we know that the gradient vector field $X = \nabla\phi$ of a first eigenfunction on S^n is a conformal vector field, i.e. X generates a 1-parameter family of conformal transformations of S^n. One can use conformal vector fields to derive certain identities for some special differential equations. Such a fact was first discovered by Pohožaev S. I. [Poh65], who made use of $X = r\frac{\partial}{\partial r}$ on $\mathbb{R}^n$. Later, Richard M. Schoen proved the following general result [Sch88].

In the next we introduce a sufficient condition for the solvability of (3.15).

Theorem 3.20 (Moser, 1973)**.** *Let g_0 be the standard metric on S^2. Suppose that $\widetilde{K} \in C^\infty(S^2)$ satisfying $\widetilde{K}(-x) = \widetilde{K}(x)$ for all $x \in S^2$, and that $\max_{S^2}\widetilde{K} > 0$. Then (3.15) has a solution $u \in C^\infty(S^2)$ with $u(-x) = u(x)$ for all $x \in S^2$.*

Proof. The idea is to apply the method of Lagrange multiplier 8.2 and the variational method, which transfers the equation to a minimizer problem.

Specifically, we set

$$\begin{aligned} \mathcal{A} &= \left\{\phi \in L_1^2(S^2) : \int_M \phi \,\mathrm{dvol} = 0 \quad \text{and} \quad \phi(-x) = \phi(x) \quad \text{a.e.}\right\} \\ \mathcal{A}_* &= \left\{\phi \in \mathcal{A} : \int_{S^2} \widetilde{K} e^{2\phi}\,\mathrm{dvol} > 0\right\} \end{aligned}$$

where $\max_{S^2}\widetilde{K} > 0$ implies $\mathcal{A}_*$ is non-empty, and set

$$J : \mathcal{A}_* \to \mathbb{R}, \quad \phi \mapsto \frac{1}{2}\|\nabla\phi\|_2^2 - 2\pi \log\int_{S^2}\widetilde{K}e^{2\phi}\,\mathrm{dvol}.$$

We consider the problem $\inf_{\mathcal{A}_*} J$.

(1) First we prove that there exists $v \in \mathcal{A}$ such that $J(v) = \inf_{\mathcal{A}_*} J$.

Since [SY94, section 5.1] claims that the constant η in (3.13) can be choosen as $1/32\pi$ for symmetric functions on (S^2, g_0),[14] we know

$$\int_{S^2} e^{2\phi}\,\mathrm{dvol} \le C\exp\left(\frac{1}{8\pi}\|\nabla\phi\|^2\right) \quad \forall\phi \in \mathcal{A},$$

and hence for any $\phi \in \mathcal{A}_*$ we have

$$\begin{aligned} J(\phi) &\ge \frac{1}{2}\|\nabla\phi\|_2^2 - 2\pi\left(\frac{1}{8\pi}\|\nabla\phi\|_2^2 + \log C + \log(\max_{S^2}\widetilde{K})\right) \\ &\ge \frac{1}{4}\|\nabla\phi\|_2^2 - 2\pi\left(\log C + \log(\max_{S^2}\widetilde{K})\right) > -\infty. \end{aligned} \tag{3.20}$$

Then we suppose (ϕ_i) is a sequence in $\mathcal{A}_*$ such that

$$J(\phi_i) \to \inf_{\mathcal{A}_*} J.$$

Note that formula (3.20) implies that $(\|\nabla\phi_i\|_2)$ is bounded. Then Poincaré inequality 10.23 implies that (ϕ_i) is bounded in $L_1^2(M)$. Since $L_1^2(M)$ is reflexive, by [Xio, theorem 3.41], there exists a subsequence, which we relabel as (ϕ_i), satisfying

$$\phi_k \rightharpoonup v \quad \text{in} \quad L_1^2(M)$$

[14] One can refer to [Mos71] for a detailed proof.

for some $v \in L_1^2(M)$. By Sobolev embedding theorem 10.13, it's easy to see that $v \in \mathcal{A}$. By remark 10.22 and corollary 3.15 we know J is weakly lower semi-continuous, and it easily follows that $v \in \mathcal{A}^*$ and

$$J(v) \leq \liminf_{i\to\infty} J(\phi_i) = \inf_{\mathcal{A}_*} J.$$

Hence we find a minimizer.

(2) Then we show that the minimizer v (up to a constant difference) is a solution to (3.15) in the sense of $L_1^2(M)$-weak solution.

Set

$$\mathcal{A}_1 = \left\{\phi \in L_1^2(M) : \int_{S^2} \widetilde{K}e^{2\phi}\,\mathrm{dvol} > 0 \quad \text{and} \quad \phi(-x) = \phi(x) \quad \text{a.e.}\right\}$$

Applying the method of Lagrange multipliers 8.2, there exists $\lambda \in \mathbb{R}$ such that v is a critical point of

$$\begin{aligned} \widetilde{J} : \mathcal{A}_1 &\to \mathbb{R}, \\ \phi &\mapsto \frac{1}{2}\|\nabla\phi\|_2^2 - \lambda\int_M \phi\,\mathrm{dvol} - 2\pi\log\int_{S^2}\widetilde{K}e^{2\phi}\,\mathrm{dvol}. \end{aligned}$$

Note that for any $\phi \in L_1^2(M)$ with $\phi(x) = \phi(-x)$ a.e., we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0}\widetilde{J}(v + t\phi) \\ &= \int_M \left(\langle\nabla v, \nabla\phi\rangle - \lambda\phi - \frac{4\pi\widetilde{K}e^{2v}}{\int_{S^2}\widetilde{K}e^{2v}\,\mathrm{dvol}}\phi\right)\mathrm{dvol} \\ &= -\int_M \left(\Delta v + \lambda + \frac{4\pi\widetilde{K}e^{2v}}{\int_{S^2}\widetilde{K}e^{2v}\,\mathrm{dvol}}\right)\phi\,\mathrm{dvol} \end{aligned}$$

Since both v and $\widetilde{K}$ are symmetric, we know v is a $L_1^2(M)$-weak solution to

$$\Delta v + \lambda + \frac{4\pi\widetilde{K}e^{2v}}{\int_{S^2}\widetilde{K}e^{2v}\,\mathrm{dvol}} = 0$$

Taking integration we know $4\pi\lambda + 4\pi = 0$, and hence $\lambda = -1$. Then setting

$$u = v + \frac{1}{2}\log\left(\frac{1}{4\pi}\int_{S^2}\widetilde{K}e^{2v}\,\mathrm{dvol}\right)$$

we know u is a $L_1^2(M)$-weak solution to (3.15).

(3) Finally we show that the solution is smooth.

By lemma 3.14, since $u \in L_1^2(M)$, we have $e^u \in L^p(M)$ for any $p \geq 1$. Then remark 10.17 (or theorem 9.24) implies that $u \in C^\infty$.

We are done. □

Corollary 3.21. *On $\mathbb{RP}^2$ with its standard metric g_0, a smooth function $\widetilde{K} \in C^\infty(\mathbb{RP}^2)$ is the Gaussian curvature of a metric $g \in \mathcal{C}_{g_0}$ iff $\widetilde{K}$ is positive at some point.*

Proof. Necessity. We lift $\widetilde{K}$ to S^2 by the canonical convering map $\pi : S^2 \to \mathbb{RP}^2$. Then Gauss-Bonnet theorem implies that $\widetilde{K}$ is positive at some point.

Sufficiency. We lift $\widetilde{K}$ to S^2 by the canonical convering map $\pi : S^2 \to \mathbb{RP}^2$. Since the standard metric on $\mathbb{RP}^2$ is also lifted to the standard metric on S^2, and the lifted function satisfies the requirements in theorem 3.20, there exists $u \in C^\infty(S^2)$ satisfying the lifted equation (3.3) and also satisfying $u(-x) = u(x)$ for all $x \in S^2$. Then clearly, u induces the solution to equation (3.3) on $\mathbb{RP}^2$. □

Remark 3.22. In fact, this result of corollary 3.21 is true for any Riemannian metric on $\mathbb{RP}^2$. (See [Aub79].)

Remark 3.23. [SY94, section 5.1] points out that if in the proof of theorem 3.20 we replace the symmetric subspace by the whole $L^2_1(S^2)$, the functional J is still bounded from below. However, it can be proved that the infimum of J can not be achieved unless $\widetilde{K}$ is constant. Therefore, when $\widetilde{K}$ does not satisfy any symmetry assumption the problem becomes much more difficult. In such a case one has to employ more complicated variational methods to obtain non-minimal critical points of J.

3.C. **Yamabe problem.** One can refer to [LP87].

4. Appendix — linear algebra

4.A. Spectral theorem.

Definition 4.1. *Let $(V, \langle \bullet, \bullet \rangle)$ be a real inner product space, and let $(E, \langle \bullet, \bullet \rangle)$ be a complex inner product space. The* ***adjoint map*** *$f^* \in \mathrm{End}(V)$ of $f \in \mathrm{End}(V)$ is given by*

$$\langle f(x), y \rangle = \langle x, f^*(y) \rangle \quad \forall x, y \in V,$$

and the ***Hermitian adjoint map*** *$\phi^* \in \mathrm{End}(E)$ of $\phi \in \mathrm{End}(E)$ is given by*

$$\langle \phi(x), y \rangle = \langle x, \phi^*(y) \rangle \quad \forall x, y \in E.$$

Moreover,[15]

(1) *The map $f \in \mathrm{End}(V)$ is* ***symmetric*** *(or* ***self-adjoint****) [resp.* ***skew-symmetric****] [resp.* ***normal****], iff $f^* = f$ [resp. $f^* = -f$] [resp. $f^* f = f f^*$], iff the corresponding (real) matrix A with respect to some orthonormal basis is* ***symmetric*** *[resp.* ***skew-symmetric****] [resp.* ***normal****], i.e. $A^T = A$ [resp. $A^T = -A$] [resp. $A^T A = A A^T$].*

(2) *The map $\phi \in \mathrm{End}(E)$ is* ***Hermitian symmetric*** *[resp.* ***Hermitian skew-symmetric****] [resp.* ***normal****], iff $\phi^* = \phi$ [resp. $\phi^* = -\phi$] [resp. $\phi^* \phi = \phi \phi^*$], iff the corresponding (complex) matrix B with respect to any Hermitian orthonormal basis is* ***Hermitian symmetric*** *[resp. Hermitian* ***skew-symmetric****] [resp.* ***normal****], i.e. $\overline{B^T} = B$ [resp. $\overline{B^T} = -B$] [resp. $\overline{B^T} B = B \overline{B^T}$].*

Here are some basic related facts:

Proposition 4.2. *(1) A real square matrix A is symmetric iff there exists $P \in O(n)$ such that $P^T A P$ is diagonal.*

(2) A complex square matrix B is normal iff there exists $Q \in U(n)$ such that $\overline{Q^T} B Q$ is diagonal.

Proof. Well-known. □

Corollary 4.3. *Let $(V, \langle \bullet, \bullet \rangle)$ be an inner product space over $\mathbb{R}$. and let $f : V \to V$ be a self-adjoint map. Then there exists an orthonormal basis $(v_i)_{i=1}^n$ of V such that*

$$f(v_i) = \lambda_i v_i \quad \text{for some } \lambda_i \in \mathbb{R}.$$

Proof. Let (e_i) be an orthonormal basis of V. Setting

$$f(e_i) = a^{ij} e_j,$$

then $A = (a^{ij})$ is a real symmetric matrix. By proposition 4.2, there exists $P \in O(n)$ such that $P^T A P$ is diagonal, there exists another orthogonal basis (v_i) such that

$$f(v_i) = \lambda_i v_i \quad \text{for some } \lambda_i \in \mathbb{R}.$$

We are done. □

Remark 4.4. For $f \in \mathrm{End}(V)$, the set of its eigenvalues is called the **spectrum** of f. Corollary 4.3 can be regarded as a so-called spectrum theorem.

[15] Let (e_i) be a basis. Then the corresponding matrix $C = (c^{ij})$ of a linear map T with respect to (e_i) is given by $T(e_i) = c^{ij} e_j$.

4.B. **Elementary symmetric functions.** The spectrum of an endomorphism is very important.[16] In the next, we introduce the elementary symmetric functions to help us analyze it.

Specifically, elementary symmetric functions can help us compute the characteristic polynomial of an endomorphism $T \in \mathrm{End}(V)$, and we have feasible algorithms for the computation.

Moreover, in subsection 5.B, we will analyze some important geometric quantities, which are related to the spectrum, via elementary symmetric functions.

Definition 4.5 (Elementary symmetric functions). *Let $(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$. We view the **elementary symmetric functions** as functions on $\mathbb{C}^n$,*

$$\sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Moreover, if $T \in \mathrm{End}(V)$ for some real vector space V, then

$$\sigma_k(T) = \sigma_k(\lambda_1, \cdots, \lambda_n) \quad \text{where } \lambda_1, \cdots, \lambda_n \text{ are the eigenvalues of } T,$$

and if $A \in M(n \times n, \mathbb{C})$, then

$$\sigma_k(A) = \sigma_k(\lambda_1, \cdots, \lambda_n) \quad \text{where } \lambda_1, \cdots, \lambda_n \text{ are the eigenvalues of } A.$$

Proposition 4.6. *It holds that*

$$\prod_{i=1}^{n} (\lambda - \lambda_i) = \sum_{k=0}^{n} (-1)^k \sigma_k(\lambda_1, \cdots, \lambda_n) \lambda^{n-k},$$

where we set $\sigma_0 = 1$. Moreover, if $A \in M(n \times n, \mathbb{C})$, then

$$\det(\lambda I - A) = \sum_{k=0}^{n} (-1)^k \sigma_k(A) \lambda^{n-k}.$$

Proof. The first assertion is trivial. The second assertion follows from the first assertion and the fact that

$$\det(\lambda I - A) = \prod_{i=1}^{n} (\lambda - \lambda_i)$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A. □

To compute $\sigma_k(A)$ directly, we introduce the following new concepts.

Definition 4.7. *Let V be a real vector space, and let $T \in \mathrm{End}(V)$. Define $\wedge^k T \in \mathrm{End}(\wedge^k V)$ on simple tensors by*

$$(\wedge^k T)(v_1 \wedge \cdots \wedge v_k) = T v_1 \wedge \cdots \wedge T v_k$$

and expand the definition linearly to all tensors. More generally, we can define $\wedge^l T^k \in \mathrm{End}(\wedge^p V)$ ($l \geq k$) on simple tensors by

$$\left(\wedge^l T^k\right)(v_1 \wedge \cdots \wedge v_l) = \sum_{1 \leq i_1 < \cdots < i_k \leq l} v_1 \wedge \cdots \wedge T v_{i_1} \wedge \cdots \wedge T v_{i_k} \wedge \cdots \wedge v_l$$

and expand the definition linearly to all tensors. If $l < k$, define $\wedge^l T^k = 0$.

[16]Some important geometric quantities are related to the spectrum. See subsection 5.A.

In particular, since $\wedge^n V = \operatorname{span}\{e_1 \wedge \cdots \wedge e_n\}$ where (e_i) is a basis of V, we can identify $\wedge^n T^k$ with the unique number $\varkappa$ satisfying[17]

$$\left(\wedge^n T^k\right)(e_1 \wedge \cdots \wedge e_n) = \varkappa(e_1 \wedge \cdots \wedge e_n).$$

Proposition 4.8. *Let V be a real vector space, and let $T \in \operatorname{End}(V)$; then*

$$\sigma_k(A) = \operatorname{tr}\left(\wedge^k T\right) = \wedge^n T^k. \tag{4.1}$$

In particular,

$$\sigma_1(A) = \operatorname{tr}(A) = \wedge^n T^1, \tag{4.2}$$

$$\sigma_n(A) = \operatorname{tr}(\wedge^n T) = \wedge^n T^n = \det T. \tag{4.3}$$

Proof. Let (e_i) be a basis of V, and we write

$$Te_i = c^{ij} e_j.$$

Then for $i_1 < \cdots < i_k$ we have

$$\begin{aligned}(\wedge^k T)(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= Te_{i_1} \wedge \cdots \wedge Te_{i_k} \\ &= c^{i_1 j_1} \cdots c^{i_k j_k} e_{j_1} \wedge \cdots \wedge e_{j_k} \\ &= \sum_{j_1 < \cdots < j_k} \varepsilon^{(j_1 \cdots j_k)}_{l_1 \cdots l_k} c^{i_1 l_1} \cdots c^{i_k l_k} e_{j_1} \wedge \cdots \wedge e_{j_k}\end{aligned}$$

and hence

$$\operatorname{tr}\left(\wedge^k T\right) = \sum_{i_1 < \cdots < i_k} \varepsilon^{(i_1 \cdots i_k)}_{l_1 \cdots l_k} c^{i_1 l_1} \cdots c^{i_k l_k}.$$

Note that

$$\begin{aligned}\left(\wedge^n T^k\right)(e_1 \wedge \cdots \wedge e_n) &= \sum_{1 \le i_1 < \cdots < i_k \le n} e_1 \wedge \cdots \wedge Te_{i_1} \wedge \cdots \wedge Te_{i_k} \wedge \cdots \wedge e_n \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} e_1 \wedge \cdots \wedge c^{i_1 j_1} e_{j_1} \wedge \cdots \wedge c^{i_k j_k} e_{j_k} \wedge \cdots \wedge e_n \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} \varepsilon^{(i_1 \cdots i_k)}_{l_1 \cdots l_k} c^{i_1 l_1} \cdots c^{i_k l_k} e_1 \wedge \cdots \wedge e_n,\end{aligned}$$

that is,

$$\wedge^n T^k = \sum_{i_1 < \cdots < i_k} \varepsilon^{(i_1 \cdots i_k)}_{l_1 \cdots l_k} c^{i_1 l_1} \cdots c^{i_k l_k}.$$

Also note that

$$\begin{aligned}\det(\lambda I - A) &= \varepsilon_{l_1 \cdots l_n}\left(\lambda \delta^{1 l_1} - c^{1 l_1}\right) \cdots \left(\lambda \delta^{n l_n} - c^{n l_n}\right) \\ &= \sum_{\alpha_1 = 1}^{2} \cdots \sum_{\alpha_n = 1}^{2} \varepsilon_{l_1 \cdots l_n} b^{1 l_1}_{\alpha_1} \cdots b^{n l_n}_{\alpha_n}\end{aligned}$$

where

$$b^{i l_i}_1 = \lambda \delta^{i l_i} \quad \text{and} \quad b^{i l_i}_2 = -c^{i l_i}.$$

[17]Clearly $\varkappa$ is independent of the choice of (e_i). This definition is well-defined.

and hence the coefficient of λ^{n-k} is

$$\sum_{I\cup J=(1\cdots n),|I|=k} \varepsilon_{l_1\cdots l_n}\delta^{j_1}_{l_{j_1}}\cdots\delta^{j_{n-k}}_{l_{j_{n-k}}} b_2^{i_1 l_{i_1}}\cdots b_2^{i_1 l_{i_1}} = \sum_{i_1<\cdots<i_k}(-1)^k\varepsilon^{(i_1\cdots i_k)}_{l_1\cdots l_k}c^{i_1 l_1}\cdots c^{i_k l_k}.$$

Then proposition 4.6 yields that

$$\sigma_k(T) = \sum_{i_1<\cdots<i_k}\varepsilon^{(i_1\cdots i_k)}_{l_1\cdots l_k}c^{i_1 l_1}\cdots c^{i_k l_k}.$$

Hence we get formula (4.1). Formula (4.2) follows directly from formula (4.1). By the above computation we know

$$\wedge^n T^n = \varepsilon_{l_1\cdots l_n}c^{1l_1}\cdots c^{nl_n} = \det(T).$$

Then formula (4.3) follows from formula (4.1). □

Remark 4.9. By the proof of we know that the entries of the corresponding matrix of $\wedge^k T$ are $k\times k$-minors of the corresponding matrix of T.

Remark 4.10. (1) *The fundamental theorem of symmetric polynomials* is another important property of elementary symmetric functions. One can refer to https://en.wikipedia.org/wiki/Elementary_symmetric_polynomial.

(2) For more properties of $\wedge^l T^k$, one can refer to https://en.wikipedia.org/wiki/Exterior_algebra.

5. APPENDIX — POINTWISE ESTIMATES OF GEOMETRIC QUANTITIES

5.A. First and second geometric quantities of symmetric (0,2)-tensors.

Let h be a symmetric $(0,2)$-tensor, and define $H \in \Gamma(M, \mathrm{End}(TM))$ by

$$h(X,Y) = \langle H(X), Y\rangle, \quad \forall X, Y \in \Gamma(M, TM).$$

Corollary 4.3 shows that h is determined by the spectrum and eigenvectors of H. Clearly, the spectrum of H has a close relation to the properties of h.

Moreover, some important geometric quantities of h are exactly given by the spectrum of H. Here we introduce the following first and second geometric quantities.

Definition 5.1. *Let h be a symmetric $(0,2)$-tensor, and define $H \in \Gamma(M, \mathrm{End}(TM))$ by*

$$h(X,Y) = \langle H(X), Y\rangle, \quad \forall X, Y \in \Gamma(M, TM).$$

By linear algebra, let $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ be the eigenvalues of H.

*(1) The k-**th trace of** h by*

$$\mathrm{tr}_k(h) = \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

*(2) The **first geometric quantity** of h is defined by*

$$G_{\min}(h) = \min_{i=1,\cdots,n} \lambda_i.$$

*(3) The **second geometric quantity of type** (n,p) of h is defined by*

$$G_{n,p}(h) = \min_{(i_1,\cdots,i_n)} \left\{ (n-p)\sum_{j=1}^{p} \lambda_{i_j} + p \sum_{j=p+1}^{n} \lambda_{i_j} \right\}$$

where $(i_1, \cdots, i_n)$ is a permutation of $(1, \cdots, n)$.

Remark 5.2. Some applications:

(1) $G_{\min}(\mathrm{Ric})$ leads to the estimates of Ric;
(2) $G_{n,p}(P)$ and the Bochner technique lead to vanishing theorems (see subsection 2.C).

Our basic idea is making estimates of $G_{\min}$ and $G_{n,p}$ via adding restrictions on $\mathrm{tr}_k(h)$.

This idea is natural. We have showed in subsection 4.B that elementary symmetric functions can help us analyze the spectrum of H.

5.B. Estimates of geometric quantities.

Definition 5.3. *Here are some basic notations:*

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\Lambda) > 0, \forall j \leq k\}, \tag{5.1}$$

$$\Lambda | j = (\lambda_1, \cdots, \widehat{\lambda_j}, \cdots \lambda_n) \quad \text{where} \quad \Lambda = (\lambda_1, \cdots, \lambda_n), \tag{5.2}$$

$$a_l = (a, \cdots, a) \in \mathbb{R}^l \quad \text{where} \quad a \in \mathbb{R}. \tag{5.3}$$

Remark 5.4. The cone Γ_k^+ represents the proper restrictions.

First, let's try to give some appropriate estimates of the first geometric quantity

$$G_{\min}(\Lambda) = \min_{i=1,\cdots,n} \lambda_i \quad \text{where} \quad \Lambda = (\lambda_1, \cdots, \lambda_n). \tag{5.4}$$

Remark 5.5 (Background)**.** Our geometric background is as following: Λ represents the spectrum of Ric, and the restriction are imposed on Schouten tensor, whose spectrum can be represented by

$$\frac{2}{n-2}\Lambda - \frac{\sigma_1(\Lambda)}{(n-1)(n-2)}1_n.$$

Therefore, we introduce the following new notations:

$$A_\Lambda = \Lambda - \frac{\sigma_1(\Lambda)}{2(n-1)}1_n \quad \text{where} \quad \Lambda = (\lambda_1, \cdots, \lambda_n), \tag{5.5}$$

and, roughly speaking, our aim is that

(1) Adding appropriate restrictions on A_Λ;
(2) Finding the lower bound of $G_{min}(\Lambda)$.

Our idea is the **continuity method**: we find a standard model case Λ_0 and A_{Λ_0}, and consider the continuous transformation

$$\Lambda_t = t\Lambda + (1-t)\Lambda_0 \quad \text{and} \quad A_t := A_{\Lambda_t} = tA_\Lambda + (1-t)A_{\Lambda_0} \quad \forall t \in [0,1].$$

If each A_t satisfies the restriction, and $G_{\min}(\Lambda_t) : [0,1] \to \mathbb{R}$ satisfies

$$G_{\min}(\Lambda_1) \geq G_{\min}(\Lambda_0),$$

then we get the conclusion.

People find the appropriate restriction as $A_\Lambda \in \overline{\Gamma}_k^+$, which has two good properties

(1) $\overline{\Gamma}_k^+$ is convex, which ensures $A_t \in \overline{\Gamma}_k^+$ for all $t \in [0,1]$;
(2) There exists Λ_0 such that $A_0 \in \overline{\Gamma}_k^+$ with $\sigma_k(A_0) = 0$. Moreover we have

$$\sigma_k(A_t) \geq 0 \implies \frac{d}{dt}\Big|_{t=0} \sigma_k(A_t) \geq 0 \implies G_{\min}(\Lambda) \geq G_{\min}(\Lambda_0).$$

Specifically, we have the following estimates.

Proposition 5.6. *Assume $k > 1$.*

(1) If $A_\Lambda \in \overline{\Gamma}_k^+$, then

$$G_{\min}(\Lambda) \geq \frac{2k-n}{2n(k-1)}\sigma_1(\Lambda). \tag{5.6}$$

(2) If $A_\Lambda \in \overline{\Gamma}_k^+$ and $k \geq n/2$, then

$$G_{\min}(\Lambda) \geq \frac{(2k-n)(n-1)}{(n-2)(k-1)}\binom{n}{k}^{-\frac{1}{k}}\sigma_k^{\frac{1}{k}}(A_\Lambda). \tag{5.7}$$

(3) If $A_\Lambda \in \overline{\Gamma}_k^+$ and $k = n/2$, then either $\lambda_i > 0$ for any i, or

$$\Lambda = (\lambda, \cdots, \lambda, 0)$$

up to a permutation. If the second case is true, then we must have $\sigma_{\frac{n}{2}}(A_\Lambda) = 0$.

Proof. Setting

$$\Lambda_0 \quad = \quad (1_{n-1}, \delta_k) \quad \text{where} \quad \delta_k = \frac{(2k-n)(n-1)}{2nk-2k-n},$$

then we have

$$A_0 = (a_{n-1}, b) \quad \text{where} \quad a = 1 - \frac{n-1+\delta_k}{2(n-1)} \quad b = \delta_k - \frac{n-1+\delta_k}{2(n-1)}$$

$$A_0 \in \overline{\Gamma}_k^+ \quad \text{and} \quad \sigma_k(A_0) = 0.$$

Suppose $\Lambda = (\lambda_1, \cdots, \lambda_n)$ and $A_\Lambda = (a_1, \cdots, a_n)$. WLOG we assume that[18]

$$\sigma_1(\Lambda) = \sigma_1(\Lambda_0) \quad \text{and} \quad \lambda_n = \min_{i=1,\cdots,n} \lambda_i.$$

Setting

$$\begin{aligned} \Lambda_t \quad &= \quad t\Lambda + (1-t)\Lambda_0, \quad \forall t \in [0,1], \\ A_t \quad &:= \quad A_{\Lambda_t} = tA_\Lambda + (1-t)A_{\Lambda_0} \quad \forall t \in [0,1], \end{aligned}$$

then we have

$$\begin{aligned} \sigma_k(A_t) \geq 0 \quad &\Longrightarrow \quad \frac{d}{dt}\Big|_{t=0} \sigma_k(A_t) \geq 0 \\ &\Longrightarrow \quad \sum_{i=1}^{n-1} (a_i - a)\sigma_{k-1}(A_0|i) + (a_n - b)\sigma_{k-1}(A_0|n) \geq 0 \\ &\Longrightarrow \quad (a_n - b)\left(\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1)\right) \geq 0 \\ &\Longrightarrow \quad a_n \geq b \quad \text{i.e.} \quad G_{\min}(\Lambda) \geq G_{\min}(\Lambda_0). \end{aligned}$$

Then formula (5.6) follows. Moreover, Maclaurin's inequality yields that

$$\sigma_1(A_\Lambda)\binom{n}{1}^{-1} \geq \sigma_k^{\frac{1}{k}}(A_\Lambda)\binom{n}{k}^{-\frac{1}{k}} \quad \forall 1 \leq k \leq n.$$

If $2k \geq n$, we have

$$G_{\min}(\Lambda) \geq \frac{2k-n}{2n(k-1)}\sigma_1(\Lambda) = \frac{(2k-n)(n-1)}{n(n-2)(k-1)}\sigma_1(A_\Lambda) \geq \frac{(2k-n)(n-1)}{(n-2)(k-1)}\binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(A_\Lambda).$$

Hence we get formula (5.7).

The proof of point (3) is omitted. One can refer to [GVW02, lemma 2]. □

Second, let's try to give some appropriate estimates of the second geometric quantity

$$\text{(5.8)} \qquad G_{n,p}(\Lambda) = \min_{(i_1,\cdots,i_n)} \left\{ (n-p)\sum_{j=1}^{p} \lambda_{i_j} + p \sum_{j=p+1}^{n} \lambda_{i_j} \right\} \quad \text{where} \quad \Lambda = (\lambda_1, \cdots, \lambda_n).$$

The geometric background of the second geometric quantity is introduced in subsection 2.C. Namely, $G_{n,p}$ is a geometric quantity arising in the Weitzenböck form for p-forms.

The process of giving the estimates of $G_{n,p}$ is similar to giving the estimates of $G_{\min}$.

[18] It's easy to see $\sigma_1(\Lambda) > 0$. Since (5.6) is invariant under the transformation from Λ to $s\Lambda$ for $s > 0$, clearly we can add the hypothesis without loss of generality.

Our idea is still the **continuity method**: we find a standard model case Λ_0, and consider the continuous transformation

$$\Lambda_t = t\Lambda + (1-t)\Lambda_0 \quad \forall t \in [0,1].$$

If each Λ_t satisfies the restriction, and $G_{n,p}(\Lambda_t) : [0,1] \to \mathbb{R}$ satisfies

$$G_{n,p}(\Lambda_1) \geq G_{n,p}(\Lambda_0) \geq 0,$$

then we get the conclusion.

Roughly speaking, people find the appropriate restriction as $\Lambda \in \overline{\Gamma}_k^+$, which has two good properties

(1) $\overline{\Gamma}_k^+$ is convex, which ensures $\Lambda_t \in \overline{\Gamma}_k^+$ for all $t \in [0,1]$;
(2) There exists Λ_0 such that $\Lambda_0 \in \overline{\Gamma}_k^+$ with $\sigma_k(\Lambda_0) = G_{n,p}(\Lambda_0) = 0$. Moreover we have

$$\sigma_k(\Lambda_t) \geq 0 \implies \frac{d}{dt}\Big|_{t=0} \sigma_k(\Lambda_t) \geq 0 \implies G_{n,p}(\Lambda) \geq G_{n,p}(\Lambda_0) = 0.$$

Specifically, we have the following estimates.

Proposition 5.7.

$$E_{n,p}^s \in \overline{\Gamma}_k^+, \quad s>0 \implies E_{n-1,p}^s \in \Gamma_{k-1}^+ \quad \textit{and} \quad E_{n-2,p-1}^s \in \Gamma_{k-1}^+, \tag{5.9}$$

$$E_{n,p} \in \overline{\Gamma}_k^+ \implies E_{n-2,p-1} \in \Gamma_k^+. \tag{5.10}$$

Proposition 5.8.

$$\begin{cases} p \leq n/2, \quad 2 \leq k, \quad 0 < s \leq 1 \\ E_{n,p}^s \in \overline{\Gamma}_k^+, \quad \sigma_k\left(E_{n,p}^s\right) = 0, \\ \Lambda \in \overline{\Gamma}_k^+ \end{cases} \implies G_{n,p}(\Lambda) \geq 0. \tag{5.11}$$

$$\begin{cases} p \leq n/2, \quad 2 \leq k, \quad 0 < s < 1 \\ E_{n,p}^s \in \overline{\Gamma}_k^+, \quad \sigma_k\left(E_{n,p}^s\right) = 0, \\ \Lambda \in \overline{\Gamma}_k^+, \quad \sigma_1(\Lambda) > 0 \end{cases} \implies G_{n,p}(\Lambda) > 0. \tag{5.12}$$

Proposition 5.9.

$$\begin{cases} 1 \leq p < n/2, \quad 2 \leq k \leq n/2 \\ E_{n,p} \in \overline{\Gamma}_k^+, \quad \sigma_k\left(E_{n,p}\right) = 0 \\ \Lambda \in \overline{\Gamma}_k^+ \end{cases} \implies G_{n,p}(\Lambda) \geq 0 \tag{5.13}$$

The equality holds if and only if $\Lambda = \mu E_{n,p}$ *for some* $\mu \geq 0$. *In particular, if* $\Lambda \in \Gamma_k^+$, *then* $G_{n,p}(\Lambda) > 0$.

Proposition 5.10.

$$\begin{cases} 2 \le p < n/2, \quad 2 \le k < n/2 \\ \sigma_k\left(E_{n,p}\right) < 0 \\ \Lambda \in \overline{\Gamma}_k^+ \end{cases} \Longrightarrow G_{n,q}(\Lambda) \ge 0 \quad \forall p \le q \le n/2 \tag{5.14}$$

$$\begin{cases} 2 \le p < n/2, \quad 2 \le k < n/2 \\ \sigma_k\left(E_{n,p}\right) < 0 \\ \Lambda \in \overline{\Gamma}_k^+, \quad \sigma_1(\Lambda) > 0 \end{cases} \Longrightarrow G_{n,q}(\Lambda) > 0 \quad \forall p \le q \le n/2 \tag{5.15}$$

$$\begin{cases} 2 \le p < n/2, \quad 2 \le k < n/2 \\ E_{n,p} \in \overline{\Gamma}_k^+, \quad \sigma_k\left(E_{n,p}\right) = 0 \\ \Lambda \in \overline{\Gamma}_k^+ \end{cases} \Longrightarrow G_{n,q}(\Lambda) \ge 0 \quad \forall p \le q \le n/2 \tag{5.16}$$

$$\begin{cases} 2 \le p < n/2, \quad 2 \le k < n/2 \\ E_{n,p} \in \overline{\Gamma}_k^+, \quad \sigma_k\left(E_{n,p}\right) = 0 \\ \Lambda \in \Gamma_k^+ \end{cases} \Longrightarrow G_{n,q}(\Lambda) > 0 \quad \forall p \le q \le n/2 \tag{5.17}$$

Proposition 5.11. *The followings are true.*

(1) $k = 2$ *and* $\frac{n}{2} \ge p \ge \left[\frac{n-\sqrt{n}}{2}\right]$*; then* $E_{n,p} \notin \Gamma_2^+$*. If* $p = \frac{n-\sqrt{n}}{2}$ *is an integer, then* $E_{n,p} \in \overline{\Gamma}_2^+$ *with* $\sigma_2\left(E_{n,p}\right) = 0$*.*

(2) $p = 2$ *and* $k \ge \left[\frac{n-\sqrt{n}}{2}\right]$*, then* $E_{n,2} \notin \Gamma_k^+$*. If* $k = \frac{n-\sqrt{n}}{2}$ *is an integer, then* $E_{n,2} \in \overline{\Gamma}_k^+$ *with* $\sigma_k\left(E_{n,2}\right) = 0$*.*

(3) For the general case, $E_{n,p} \notin \overline{\Gamma}_k^+$*, if* $3 \le p \le n/2$*, and*

$$k \ge \frac{n-2p+4-\sqrt{n-2p+4}}{2};$$

or if $3 \le k < n/2$*, and*

$$p \ge \frac{n-k+2-\sqrt{n-k+2}}{2}.$$

In particular, if $n > 4$ *and* $k = \left[\frac{n+1}{2}\right] + 1 - p$*, then* $E_{n,p} \notin \overline{\Gamma}_k^+$*.*

One can refer to [GLW05] for their proofs.

6. Appendix — geometry

6.A. **Schur theorem.**

Theorem 6.1 (Schur)**.** *Let (M, g) be a Riemannian manifold with dimension $n \geq 3$. If*

$$R = fg \owedge g$$

then f is constant.

Proof. By definition 1.5, we know

$$R_{ijkl} = f\left(g_{il}g_{jk} - g_{ik}g_{jl}\right).$$

Therefore,

$$R_{ijkl;s} = (\partial_s f)\left(g_{il}g_{jk} - g_{ik}g_{jl}\right).$$

The third Bianchi identity says that

$$R_{ijkl;s} + R_{ijls;k} + R_{ijsk;l} = 0.$$

Hence

$$(\partial_s f)\left(g_{il}g_{jk} - g_{ik}g_{jl}\right) + (\partial_k f)\left(g_{is}g_{jl} - g_{il}g_{js}\right) + (\partial_l f)\left(g_{ik}g_{js} - g_{is}g_{jk}\right) = 0$$

that is,

$$\det\begin{pmatrix} \partial_s f & \partial_k f & \partial_l f \\ g_{is} & g_{ik} & g_{il} \\ g_{js} & g_{jk} & g_{jl} \end{pmatrix} = 0, \quad \forall i, j, k, l, s.$$

Since $n \geq 3$, the subsequent lemma 6.3 yields $\partial_s f = 0$ for all s. Hence f is constant. □

Remark 6.2. By proposition 1.6 we know

$$\mathrm{Ric} = (n-1)fg \quad \text{and} \quad \mathrm{scal} = (n-1)nf.$$

Moreover, the sectional curvature $K = f$, since

$$R(X, Y, Y, X) = fg \owedge g(X, Y, Y, X) = f\left(|X|^2|Y|^2 - \langle X, Y\rangle^2\right).$$

Lemma 6.3. *Let H be an inner product space over $\mathbb{R}$ with $\dim H \geq 3$. Let $(e_i)_{i=1}^n$ be a basis of H, and let $g_{ij} = \langle e_i, e_j\rangle$. If $a, b, c \in \mathbb{R}$ satisfy*

$$\det\begin{pmatrix} a & b & c \\ g_{i1} & g_{i2} & g_{i3} \\ g_{j1} & g_{j2} & g_{j3} \end{pmatrix} = 0, \quad \forall i, j$$

then $a = b = c = 0$.

Proof. Note that the **Gram matrix** with respect to e_1, e_2 and e_3

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

is positive definite since

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \langle xe_1 + ye_2 + ze_3, xe_1 + ye_2 + ze_3 \rangle .$$

Now set

$$\alpha_i = \begin{pmatrix} g_{i1} & g_{i2} & g_{i3} \end{pmatrix} \quad i = 1, 2, 3.$$

Then $(\alpha_i)_{i=1}^3$ is a basis of $\mathbb{R}^3$. The condition says that

$$\begin{pmatrix} a & b & c \end{pmatrix} \in \operatorname{span}\{\alpha_1, \alpha_2\} \bigcap \operatorname{span}\{\alpha_1, \alpha_3\} \bigcap \operatorname{span}\{\alpha_2, \alpha_3\}$$

Clearly, $a = b = c = 0$, since the unique expression

$$\begin{pmatrix} a & b & c \end{pmatrix} = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$$

must have coefficients $c_1 = c_2 = c_3 = 0$. □

6.B. **Derivations — (1,1)-tensors, curvature derivations.**

Definition 6.4. *A map $T \mapsto DT$ on tensors is called a* ***derivation*** *if it preserves the type of the tensor; is linear; commutes with contractions; and satisfies the product rule*

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes DT_2.$$

The curvature derivation is an important example:

Proposition 6.5. *The operator $R_{X,Y}$ is a derivation for any $X, Y \in \Gamma(M, TM)$.*

Proof. It's clear that $R_{X,Y}$ preserves the type of the tensor; is linear; commutes with contractions (∇_Z commutes with contraction). It suffices to show that $R_{X,Y}$ satisfies the product rule. Note that

$$\nabla_X\nabla_Y(\theta_1 \otimes \theta_2) = (\nabla_X\nabla_Y\theta_1) \otimes \theta_2 + (\nabla_Y\theta_1) \otimes \nabla_X\theta_2 + (\nabla_X\theta_1) \otimes \nabla_Y\theta_2 + \theta_1 \otimes \nabla_X\nabla_Y\theta_2$$

It follows that

$$[\nabla_X, \nabla_Y](\theta_1 \otimes \theta_2) = ([\nabla_X, \nabla_Y]\theta_1) \otimes \theta_2 + \theta_1 \otimes [\nabla_X, \nabla_Y]\theta_2.$$

Then we know $R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ satisfies the product rule. □

Remark 6.6. Note that

$$R_{X,Y}T = \left(\nabla^2 T\right)(X, Y) - \left(\nabla^2 T\right)(Y, X).$$

It follows that

$$R_{X,Y}f = 0 \quad \forall f \in C^\infty(M) \tag{6.1}$$

since Hess$f$ is symmetric.

Proposition 6.7. *Let D be a derivation. Then*

$$D(\omega \wedge \eta) = (D\omega) \wedge \eta + \omega \wedge D\eta, \quad \forall \omega \in \Omega^k(M),\ \eta \in \Omega^l(M).$$

Proof. Recall that

$$\operatorname{Alt}\alpha = \frac{1}{k!}\sum_\pi (-1)^{|\pi|}\left({}^\sigma\alpha\right) \quad \text{where} \quad {}^\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \tag{6.2}$$

and that

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega \otimes \eta), \quad \forall \omega \in \Omega^k(M),\ \eta \in \Omega^l(M). \tag{6.3}$$

It follows that

$$\begin{aligned} D(\omega \wedge \eta) &= \frac{1}{k!l!}\sum_{\sigma}(-1)^{|\sigma|}D\left({}^{\sigma}(\omega \otimes \eta)\right), \\ (D\omega)\wedge \eta &= \frac{1}{k!l!}\sum_{\sigma}(-1)^{|\sigma|}\left({}^{\sigma}\left((D\omega)\otimes \eta\right)\right), \\ \omega \wedge D\eta &= \frac{1}{k!l!}\sum_{\sigma}(-1)^{|\sigma|}\left({}^{\sigma}\left(\omega \otimes D\eta\right)\right). \end{aligned}$$

It suffices to show the stronger claim:

$$D\left({}^{\sigma}(T_1 \otimes T_2)\right) = {}^{\sigma}\left((DT_1)\otimes T_2\right) + {}^{\sigma}\left(T_1 \otimes DT_2\right) \quad \forall T_1, T_2.$$

By linearity, WLOG we assume that $T_1 = \gamma_1 \otimes \cdots \gamma_k$ and $T_2 = \gamma_{k+1} \otimes \cdots \otimes \gamma_{k+l}$. Seeting $\pi = \sigma^{-1}$, then

$$D\left({}^{\sigma}(T_1 \otimes T_2)\right) = D\left(\gamma_{\pi(1)} \otimes \cdots \otimes \gamma_{\pi(n)}\right) = \sum_{i=1}^{n}\gamma_{\pi(1)} \otimes \cdots \otimes \left(D\gamma_{\pi(i)}\right) \otimes \cdots \otimes \gamma_{\pi(n)},$$

and

$$\begin{aligned} {}^{\sigma}\left((DT_1)\otimes T_2\right) + {}^{\sigma}\left(T_1 \otimes DT_2\right) &= {}^{\sigma}\left((DT_1)\otimes T_2 + T_1 \otimes DT_2\right) = {}^{\sigma}\left(D(T_1 \otimes T_2)\right) \\ &= {}^{\sigma}\left(\sum_{i=1}^{n}\gamma_1 \otimes \cdots \otimes (D\gamma_i) \otimes \cdots \otimes \gamma_n\right) \\ &= \sum_{i=1}^{n}\gamma_{\pi(1)} \otimes \cdots \otimes \left(D\gamma_{\pi(\sigma(i))}\right) \otimes \cdots \otimes \gamma_{\pi(n)} \\ &= \sum_{i=1}^{n}\gamma_{\pi(1)} \otimes \cdots \otimes \left(D\gamma_{\pi(i)}\right) \otimes \cdots \otimes \gamma_{\pi(n)}. \end{aligned}$$

We are done. □

Remark 6.8. Here we use the definitions (6.2) and (6.3). We don't say that $D\omega$ must be a differential form.

One should note that any derivation is determined by Df and DX:

Proposition 6.9. *Let $T \mapsto DT$ be a derivation. If we know Df for all $f \in C^\infty(M)$ and know DX for all $X \in \Gamma(M, TM)$, then we know DT for all tensor T.*

Proof. By linearity and the product rule, it suffices to show that we know $D\omega$ for all $\omega \in \Gamma(M, T^*M)$. Note that

$$D\left(C\left(X \otimes \omega\right)\right) = C\left(D\left(X \otimes \omega\right)\right) = C\left((DX)\otimes \omega\right) + C\left(X \otimes D\omega\right)$$

where C is the contraction. Hence we know $C\left(X \otimes D\omega\right)$ for any $X \in \Gamma(M, TM)$ and for any $\omega \in \Gamma(M, T^*M)$. Let (E_i) be a local frame, and let (E^i) be its dual. Then

$$C\left(E_i \otimes D\omega\right) = E_i(E^j)\cdot(D\omega)(E_j) = (D\omega)\left(E_i\right).$$

Hence we know $D\omega$ for any $\omega \in \Gamma(M, T^*M)$. □

Conversely, we want to construct a derivation by Df and DX. But this is not easy, since except the linearity, the derivation D satisfies $D(fX) = Df \cdot X + f \cdot DX$. However, we have a easy model that $Df = 0$ and D is tensorial:

Proposition 6.10. *For any* $(1,1)$*-tensor* L*, which is also regarded as an element in* $\Gamma(M, \mathrm{End}(TM))$*, there exists a unique derivation* D *such that* $Df = 0$ *for all* $f \in C^\infty(M)$ *and* $DX = LX$ *for any* $X \in \Gamma(M, TM)$*.*

Proof. See [Pet16, section 2.3.1] for the existence of such D. The uniqueness follows from proposition 6.9. □

Remark 6.11. For convenience, we write $L = D$. For any $(0,k)$-tensor T, we have

$$(LT)(X_1, \cdots, X_k) = -\sum T(X_1, \cdots, LX_i, \cdots, X_k) \tag{6.4}$$

Remark 6.12. Since L is tensorial, we can consider L pointwisely. One should note that L_p coincides with the **action of an endomorphisms on tensors** [Pet16, section 2.3.1].

An important thing is that we can study the properties of $R_{X,Y}$ via such derivations of $(1,1)$-tensors, since $R_{X,Y}$, as a derivation, is just the induced derivation by itself as a $(1,1)$-tensor. (This accords with the fact that $R_{X,Y}f = 0$ for all $f \in C^\infty(M)$.)

Proposition 6.13. *If there exists* $(1,1)$*-tensor* L *such that*

$$R_{X,Y}Z = LZ \quad \forall Z \in \Gamma(M, TM),$$

then $R_{X,Y} = L$ *as derivations. In fact, such* L *uniquely exists, and* $L = R_{X,Y}$ *as* $(1,1)$*-tensors.*

Proof. Since $R_{X,Y}$ is a derivation and $R_{X,Y}f = 0$ for any $f \in C^\infty(M)$ (see proposition 6.5 and remark 6.6), the conclusion follows from proposition 6.9. □

The above proposition gives us a new perspective to deal with curvature tensors; that is, we regard it as a derivation induced by a $(1,1)$-tensor.

Finally, we introduce some basic properties of L.

Proposition 6.14. *If* L *is skew-symmetric, then we have*

$$\langle LT, S\rangle = -\langle LS, T\rangle \quad \forall S, T \in \Gamma(M, (\otimes^s TM) \otimes (\otimes^t T^*M))$$

Proof. By the product rule, it suffices to verify it on $\Gamma(M, T^*M)$ (on $\Gamma(M, TM)$ it holds by condition). Let (E_i) be a local orthonormal frame. Then for $\omega, \eta \in \Gamma(M, T^*M)$ we have

$$\begin{aligned}
\langle L\omega, \eta\rangle &= \langle (L\omega)(E_i), \eta(E_i)\rangle = -\langle \omega(LE_i), \eta(E_i)\rangle \\
&= -\left\langle LE_i, E_j\right\rangle \left\langle \omega(E_j), \eta(E_i)\right\rangle = \left\langle LE_j, E_i\right\rangle \left\langle \omega(E_j), \eta(E_i)\right\rangle \\
&= \left\langle \omega(E_j), \eta(LE_j)\right\rangle = -\left\langle \omega(E_j), (L\eta)(E_j)\right\rangle = -\langle \omega, L\eta\rangle .
\end{aligned}$$

We are done. □

For general properties of L, one can refer to [Pet16, section 2.3.1].

6.C. **Ricci idendity.**

Theorem 6.15 (Ricci identity for covariant derivatives)**.** *Let* (M,g) *be a Riemannian manifold. For the tensor*

$$T = T_{i_1\cdots i_r}^{j_1\cdots j_s} dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}},$$

we have the following Ricci identity:

$$\nabla_k \nabla_l T_{i_1\cdots i_r}^{j_1\cdots j_s} - \nabla_l \nabla_k T_{i_1\cdots i_r}^{j_1\cdots j_s} = \sum_{m=1}^{s} R_{klp}{}^{j_m} T_{i_1\cdots i_r}^{j_1\cdots j_{m-1} p j_{m+1}\cdots j_s} - \sum_{t=1}^{r} R_{kli_t}{}^{q} T_{i_1\cdots i_{t-1} q i_{t+1}\cdots i_r}^{j_1\cdots j_s}.$$

In particular, one has

$$\nabla_k \nabla_l X^i - \nabla_l \nabla_k X^i = R_{klp}{}^{i} X^p$$

and

$$\nabla_k \nabla_l \eta_i - \nabla_l \nabla_k \eta_i = -R_{kli}{}^{s} \eta_s.$$

Proof. We use the following convention in this problem:

$$(\square) = \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, dx^{j_1}, \cdots, dx^{j_s}\right).$$

Then it follows from proposition 6.5 and formula (6.1) that

$$\begin{aligned}
&\nabla_k \nabla_l T_{i_1\cdots i_r}^{j_1\cdots j_s} - \nabla_l \nabla_k T_{i_1\cdots i_r}^{j_1\cdots j_s} \\
= &\left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} T\right)(\square) - \left(\nabla_{\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l}} T\right)(\square) - \left(\nabla_{\frac{\partial}{\partial x^l}} \nabla_{\frac{\partial}{\partial x^k}} T\right)(\square) + \left(\nabla_{\nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^k}} T\right)(\square) \\
= &\left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} T - \nabla_{\frac{\partial}{\partial x^l}} \nabla_{\frac{\partial}{\partial x^k}} T - \nabla_{\left[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right]} T\right)(\square) \\
= &\left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\left(T_{q_1\cdots q_r}^{p_1\cdots p_s} dx^{q_1} \otimes \cdots \otimes dx^{q_r} \otimes \frac{\partial}{\partial x^{p_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{p_s}}\right)\right)(\square) \\
= &\sum_{t=1}^{r}\left(T_{q_1\cdots q_r}^{p_1\cdots p_s} dx^{q_1} \otimes \cdots \otimes R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) dx^{q_t} \otimes \cdots \otimes dx^{q_r} \otimes \frac{\partial}{\partial x^{p_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{p_s}}\right)(\square) \\
&+ \sum_{m=1}^{s}\left(T_{q_1\cdots q_r}^{p_1\cdots p_s} dx^{q_1} \otimes \cdots \otimes dx^{q_r} \otimes \frac{\partial}{\partial x^{p_1}} \otimes \cdots \otimes R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) \frac{\partial}{\partial x^{p_m}} \otimes \cdots \otimes \frac{\partial}{\partial x^{p_s}}\right)(\square) \\
= &-\sum_{t=1}^{r}\left(T_{q_1\cdots q_r}^{p_1\cdots p_s} dx^{q_1} \otimes \cdots \otimes R_{kli}{}^{q_t} dx^{i} \otimes \cdots \otimes dx^{q_r} \otimes \frac{\partial}{\partial x^{p_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{p_s}}\right)(\square) \\
&+ \sum_{m=1}^{s}\left(T_{q_1\cdots q_r}^{p_1\cdots p_s} dx^{q_1} \otimes \cdots \otimes dx^{q_r} \otimes \frac{\partial}{\partial x^{p_1}} \otimes \cdots \otimes R_{klp_m}{}^{j} \frac{\partial}{\partial x^{j}} \otimes \cdots \otimes \frac{\partial}{\partial x^{p_s}}\right)(\square) \\
= &\sum_{m=1}^{s} R_{klp}{}^{j_m} T_{i_1\cdots i_r}^{j_1\cdots j_{m-1} p j_{m+1}\cdots j_s} - \sum_{t=1}^{r} R_{kli_t}{}^{q} T_{i_1\cdots i_{t-1} q i_{t+1}\cdots i_r}^{j_1\cdots j_s}.
\end{aligned}$$

We are done. ∎

6.D. **A brief introduction to Bochner technique.** We start with a basic computation.

Lemma 6.16. *For a general tensor s, we have*

$$\Delta \frac{1}{2}|s|^2 = |\nabla s|^2 - g\left(\nabla^* \nabla s, s\right). \tag{6.5}$$

Proof. Recall the basic formula

$$X\langle T_1, T_2\rangle = \langle \nabla_X T_1, T_2\rangle + \langle T_1, \nabla_X T_2\rangle \quad \text{if } T_1 \text{ and } T_2 \text{ are tensors of the same type.} \tag{6.6}$$

Then given any local coordinates (x^i), it follows that

$$\begin{aligned}\Delta \frac{1}{2}|s|^2 &= \frac{1}{2} g^{ij} \nabla_i \nabla_j \langle s, s\rangle = g^{ij} \nabla_i \left\langle \nabla_j s, s\right\rangle \\ &= \left\langle g^{ij} \nabla_i \nabla_j s, s\right\rangle + g^{ij}\left\langle \nabla_j s, \nabla_i s\right\rangle = |\nabla s|^2 - g\left(\nabla^* \nabla s, s\right).\end{aligned}$$

We are done. □

Then our basic ideas are as follows.

Lemma 6.17. *Tensor T if parallel if it satisfies the following two conditions:*
(1) $|T|$ admits its maximum at some point;
(2) $g\left(\nabla^* \nabla T, T\right) \leq 0$.

Proof. It follows from formula (6.5) and the maximum principle [Pet16, theorem 7.1.7]. □

Definition 6.18. *Define the **Weitzenböch curvature operator*** Ric *on* $\Gamma(M, \otimes^k T^* M)$ *by*[19]

$$\operatorname{Ric}(T)\left(X_1, \cdots, X_k\right) = \sum\left(R(E_j, X_i) T\right)\left(X_1, \cdots, E_j, \cdots, X_k\right)$$

*where (E_i) is a local orthonormal frame. Then we define the **Lichnerowicz Laplacian** Δ_L on $\Gamma(M, \otimes^k T^* M)$ by*

$$\Delta_L T = \nabla^* \nabla T + c \operatorname{Ric}(T)$$

*for a suitable constant $c > 0$. The **Hodge Laplacian** Δ_H is of this type with $c = 1$.*

Lemma 6.19. *Given $T \in \Gamma(M, \otimes^k T^* M)$. If T satisfies*

$$\Delta_L T = 0 \quad \text{and} \quad g(\operatorname{Ric}(T), T) \geq 0, \tag{6.7}$$

and if $|T|$ admits its maximum at some point, then T is parallel.

Proof. This follows from the preceding lemma and the definition of Δ_L. □

If a tensor T satisfies $\Delta_L = 0$, T may represent some property of M, such as the topology property. If we add some constraints to the curvatures, $g(\operatorname{Ric}(T), T)$ may hold. Thus we can apply the Bochner technique to T,[20] and then derive some vanishing theorems or other estimates.

6.E. **Flat manifolds.**

Definition 6.20. *A (pseudo-)Riemannian manifold is called **flat** if is it locally isometric to a (pseudo-)Euclidean space.*

Proposition 6.21. *A (pseudo-)Riemannian manifold M if flat iff $R = 0$.*

[19] We use the "Ric" notation since $\operatorname{Ric}(\omega)(X) = \omega(\operatorname{Ric}(X))$.
[20] For the good case that M is compact, $|T|$ always admits its maximum.

Proof. If M is flat, clearly $R = 0$. In the next we suppose that (M, g) has vanishing curvature tensor.

First, we show that g shares one important property with (pseudo-)Euclidean spaces: it admits a parallel orthonormal frame (E_i) in a neighborhood U of each point. Let $p \in M$, and choose an orthonormal basis (e_i) of T_pM. By solving ODEs, there exist a local frame (E_i) such that $E_i|_p = e_i$ for each i. Bacause parallel transport preserves inner products (see [Lee18, proposition 5.5]), the frame (E_i) is orthonormal.

Second, we show that (E_i) induces the desired coordinates. Since each E_i is parallel on U, we have

$$[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = 0 \quad \forall i, j \quad \text{on } U,$$

and hence by [Lee13, theorem 9.46], there exist local coordinates (x^i) on $V \subset U$ such that

$$E_i = \frac{\partial}{\partial x^i} \quad \forall i \quad \text{on } V.$$

Clearly, (x^i) gives the local isometry. □

Problem 6.22. *Being flat is special, and having constant curvature is special. How about the general cases? To what extent does curvature R determine the Riemannian metric g?*

One can refer to [Yau74] for the above problem.

6.F. **Hodge theorem.**

Theorem 6.23 (Hodge)**.** *Let (M, g) be a closed Riemannian manifold. Then* $\dim \mathcal{H} < \infty$*, and there exists a bounded linear operator $G : \Omega^*(M) \to \Omega^*(M)$ with*

(1) $\ker G = \mathcal{H}$*;*

(2) $G(\Omega^q(M)) \subset \Omega^q(M)$*, and*

$$G * = * G, \quad Gd = dG, \quad G\delta = \delta G.$$

(3) G is compact with respect to the norm $(\bullet, \bullet)$*;*

(4) For any $\omega \in \Omega^*(M)$ we have

$$\omega = \omega_h + \Delta_H (G\omega)$$

where $\omega_h \in \mathcal{H}$.

Proof. See [Mei13, theorem 5.2.12]. □

Corollary 6.24. *Let (M, g) be a closed Riemannian manifold. If $K \in C^\infty(M)$ satisfies* $\int_M K\,\mathrm{dvol} = 0$*, then there exists $f \in C^\infty(M)$ with $\Delta f = K$.*

Proof. By Hodge theorem 6.23, $K = K_h + \Delta f$ for some harmonic function K_h and some $f \in C^\infty(M)$. By the maximum principle, the harmonic function K_h is constant. Taking integration we get $\int_M K_h\,\mathrm{dvol} = 0$, it follows that $K_h = 0$. □

7. Appendix — first order PDEs

7.A. Frobenius theorem.

Definition 7.1. *Some basic concepts for* $\Gamma(M, TM)$*:*

(1) A ***(smooth) distribution*** *on* M *of rank* k *is a rank-*k *(smooth) subbundle of* TM*;*
(2) We say that a distribution D *is* ***involutive*** *if given any pair of smooth local sections of* D*,* $[X, Y]$ *is also a local section of* D*;*
(3) For a distribution D*, a non-empty immersed submanifold* $N \subset M$ *is called an* ***integral manifold of*** D *if* $T_pN = D_p$ *at each point* $p \in N$*;*
(4) A distribution D *on* M *is said to be* ***integrable*** *if each point of* M *is contained in an integral manifold* D*;*
(5) For a rank-k *distribution* $D \subset TM$*, we say that a smooth coordinate chart* (U, φ) *on* M *is* ***flat for*** D *if* $\varphi(U)$ *is a cube in* $\mathbb{R}^n$*, and at points of* U*,* D *is spanned by the first* k *coordinate vector fields* $\partial/\partial x^1, \cdots, \partial/\partial x^k$*;*
(6) A distribution D *is said to be* ***completely integrable*** *if there exists a flat chart for* D *in a neighborhood of each point of* M*.*

Some basic concepts for $\Gamma(M, T^*M)$*:*

(1) If D *is a rank-*k *distribution, any* $n-k$ *linearly independent* 1*-forms* $\omega^1, \cdots, \omega^{n-k}$ *defined on an open set* U *are said to be* ***local defining forms for*** D *if*

$$D_q = \ker \omega^1|_q \cap \cdots \cap \ker \omega^{n-k}|_q \quad \forall q \in U.$$

(2) Let D *be a rank-*k *distribution. We say that* $\omega \in \Omega^l(M)$ ***annihilates*** D *if*

$$\omega(X_1, \cdots, X_l) = 0 \quad \text{whenever } X_1, \cdots, X_l \text{ are local sections of } D.$$

Remark 7.2. Sometimes we may construct some special functions via 1-forms. By Poincaré lemma, $Xf = g$ can be reduced to that $X(\omega) = g$ and $d\omega = 0$.

Remark 7.3. Clearly for a distribution we know

$$\text{completely integrable} \implies \text{integrable} \implies \text{involutive}.$$

Lemma 7.4 (Local coframe criterion for involutivity)**.** *Let* D *be a smooth distribution of rank* k *on a smooth* n*-manifold* M*, and let* $U \subset M$ *be an open subset. TFAE:*

(1) D *is involutive on* U*.*
(2) All (or some) local defining forms $\omega^1, \cdots, \omega^{n-k}$ *for* D*, which are defined on* U*, satisfy that* $d\omega^1, \cdots, d\omega^{n-k}$ *annihilates* D*.*
(3) All (or some) local defining forms $\omega^1, \cdots, \omega^{n-k}$ *for* D*, which are defined on* U*, satisfy that there exist* 1*-forms* $\{\alpha^i_j : i, j = 1, \cdots, n-k\}$ *with*

$$d\omega^i = \sum_{j=1}^{n-k} \omega^j \wedge \alpha^i_j.$$

Proof. (1) $\iff$ (2) easily follows from the formula

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

(2) $\iff$ (3) follows from [Lee13, lemma 19.6]. □

Definition 7.5. *Any subbundle $A \subset T^*M$ is called* ***integrable*** *if for any local frame $(\omega^1, \cdots, \omega^l)$ of A, there exist 1-forms $\{\alpha^i_j : i, j = 1, \cdots, l\}$ with*

$$d\omega^i = \sum_{j=1}^{l} \omega^j \wedge \alpha^i_j$$

which is denoted by

$$d\omega^i \equiv 0 \mod (\omega^1, \cdots, \omega^l).$$

Corollary 7.6. *For any distribution D, the colletion*

$$A = \bigcup_{p\in M} A_p \quad \text{where} \quad A_p = \left\{\eta \in T^*_pM : \eta(v) = 0, \forall v \in D_p\right\}.$$

is a subbundle of T^*M. Conversely, for any subbundle $A \subset T^*M$, the colletion

$$D = \bigcup_{p\in M} D_p \quad \text{where} \quad D_p = \left\{v \in T_pM : \eta(v) = 0, \forall \eta \in A_p\right\}.$$

is a distribution. Moreover, A is integrable iff D is involutive.

Proof. Given a distribution D, let $(X_i)^k_{i=1}$ be a local frame of D on U. In a neighborhood of each point in U we can complete the k-tuple $(X_i)^k_{i=1}$ to a smooth local frame $(X_i)^n_{i=1}$ of TM by [Lee13, proposition 10.15]. Let $(\omega^i)^n_{i=1}$ be the dual of $(X_i)^n_{i=1}$. Then $(\omega^{k+1}, \cdots, \omega^n)$ is a local frame of A. Then by the local frame criterion ([Lee13, lemma 10.32]) for subbundles, A forms a subbundle.

Similarly given a subbundle $A \subset T^*M$, D forms a distribution.

The final assertion then follows from lemma 7.4. □

Theorem 7.7 (Frobenius theorem)**.** *Every involutive distribution is completely integrale.*

Proof. See [Lee13, theorem 19.12]. □

Corollary 7.8. *Let $A \subset T^*M$ be an integrable subbundle of rank r. Then for any $p \in M$, there exist a neighborhood U and smooth functions $f^1, \cdots, f^r$ on U such that A is locally spanned by $(df^i)^r_{i=1}$.*

Proof. By corollary 7.6, the colletion

$$D = \bigcup_{p\in M} D_p \quad \text{where} \quad D_p = \left\{v \in T_pM : \eta(v) = 0, \forall \eta \in A_p\right\}.$$

is the corresponding involutive distribution, and hence there exist local coordinates (x^i) on U such that

$$\text{span}\left\{\left.\frac{\partial}{\partial x^1}\right|_q, \cdots, \left.\frac{\partial}{\partial x^k}\right|_q\right\} = D_q \quad \forall q \in U.$$

Then clearly we have

$$A_q = \text{span}\left\{dx^{k+1}|_q, \cdots, dx^n|_q\right\} \quad \forall q \in U$$

We are done. □

Remark 7.9. In other words, we have the following conclusion:

Let $\omega^1, \cdots, \omega^r$ be smooth 1-forms defined on a neighborhood U of p in M. If $\omega^1, \cdots, \omega^r$ are linearly independent on U and if [21]

$$d\omega^j \equiv 0 \mod (\omega^1, \cdots, \omega^r),$$

then there exist a smaller neighborhood V of p and smooth functions $f^1, \cdots, f^r$ on V such that

$$\omega^j = \sum_{k=1}^{r} h^j_k df^k \quad \text{for } j = 1, \cdots, r,$$

where each h^j_k is a smooth function on V.

7.B. **First order PDEs.** First, we review Frobenius theorem from the perspective of PDE. Given $p \in U$ and given a local chart $(U, (x^i))$ centered at p. Suppose that the involutive distribution D is locally spanned by $X_1, \cdots, X_k$ where

$$X_i = a^j_i \frac{\partial}{\partial x^j} \quad \forall 1 \le i \le k.$$

If $(V, (u^i))$ is a flat chart for D near p, then $X_i = \frac{\partial}{\partial u^i}$, and hence

$$X_i(u^l) = a^j_i \frac{\partial u^l}{\partial x^j} = 0 \quad \forall 1 \le i \le k, \quad \forall l > k.$$

Therefore, we know the following PDEs (where $X_1, \cdots, X_m$ are linearly independent and satisfy the compatibility condition of being involutive)

$$\begin{cases} X_1(u) = a^j_1 \dfrac{\partial u}{\partial x^j} = 0 \\ \cdots \\ X_k(u) = a^j_k \dfrac{\partial u}{\partial x^j} = 0 \end{cases} \tag{7.1}$$

have local solutions $u^{k+1}, \cdots, u^n$ such that $\nabla u^{k+1}, \cdots, \nabla u^n$ are linearly independent.

More generally, we consider the non-homogeneous cases. I.e. we solve the first order non-homogeneous overdetermined linear PDEs.

Proposition 7.10. *Let $W \subset \mathbb{R}^n$ be an open subset, and let $X_1, \cdots, X_m$ be linearly independent smooth vector fields on W. Suppose that there are $c^k_{ij}, f_l \in C^\infty(W)$ for $1 \le i, j, k, l \le m$, such that the following compatibility conditions are satisfied:*

$$[X_i, X_j] = c^k_{ij} X_k, \tag{7.2}$$

$$X_i f_j - X_j f_i = c^k_{ij} f_k. \tag{7.3}$$

Then for each $p \in W$, the following PDEs

$$\begin{cases} X_1(u) = a^j_1 \dfrac{\partial u}{\partial x^j} = f_1 \\ \cdots \\ X_m(u) = a^j_m \dfrac{\partial u}{\partial x^j} = f_m \end{cases} \tag{7.4}$$

[21] The following integrability condition holds under invertible transformations and hence is a constraint for the local subbundle.

have local solutions near each point in W.

Moreover, suppose we are given an embedded codimension-m submanifold $S \subset W$ such that T_pS is complementary to the span of $(A_i)_{i=1}^m$ at each $p \in S$. Then for each $p \in S$, there is a neighborhood U of p such that for every $\varphi \in C^\infty(S \cap U)$, there exists a unique solution $u \in C^\infty(U)$ to the following overdetermined Cauchy problem:

$$\begin{cases} X_i(u) = a_i^j \dfrac{\partial u}{\partial x^j} = f_i, & \forall 1 \le i \le m, \\ u|_{S\cap U} = \varphi. \end{cases} \tag{7.5}$$

Remark 7.11. Compatibility condition (7.2) exactly means that $(X_1, \cdots, X_m)$ is an involutive distribution; compatibility condition (7.3) is the natural added constraint for non-homogeneous cases, since

$$\begin{aligned} [X_i, X_j]u &= c_{ij}^k X_k u = c_{ij}^k f_k \\ &= X_i X_j u - X_j X_i u = X_i f_j - X_j f_i. \end{aligned}$$

Proof. The idea is as follows: via Frobenius theorem, find ω such that

$$\omega(A_i) = f_i \quad \forall 1 \le i \le m,$$

and show that ω has some **closedness** to a certain degree; then ω will induce u just like what we do for proving Poincaré lemma.

We find a flat chart $\left(U, (v, w) = (v^1, \cdots, v^m, w^1, \cdots, w^{n-m})\right)$ for D centered at p by Frobenius theorem 7.7, so

$$\operatorname{span}\{X_1, \cdots, X_m\} = \operatorname{span}\left\{\frac{\partial}{\partial v^1}, \cdots, \frac{\partial}{\partial v^m}\right\}.$$

Note that $(X_1, \cdots, X_m, \partial/\partial w^1, \cdots, \partial/\partial w^{n-m})$ is a local frame of TM on U; then let $(\alpha^1, \cdots, \alpha^m, \beta^1, \cdots, \beta^{n-m})$ be its dual. Setting

$$\omega = \omega_k dv^k := f_k \alpha^k,$$

and hence

$$\omega(X_i) = f_i \quad \forall 1 \le i \le m, \quad \text{and} \quad \omega\left(\frac{\partial}{\partial w^j}\right) = 0 \quad \forall 1 \le j \le n-m. \tag{7.6}$$

Moreover, we have

$$\begin{aligned} d\omega\left(X_i, X_j\right) &= X_i\left(\omega(X_j)\right) - X_j\left(\omega(X_i)\right) - \omega\left([X_i, X_j]\right) \\ &= X_i f_j - X_j f_i - \omega\left(c_{ij}^k X_k\right) \\ &= X_i f_j - X_j f_i - c_{ij}^k f_k = 0. \end{aligned}$$

It follows that

$$0 = d\omega\left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j}\right) = \frac{\partial}{\partial v^i}\left(\omega\left(\frac{\partial}{\partial v^j}\right)\right) - \frac{\partial}{\partial v^j}\left(\omega\left(\frac{\partial}{\partial v^i}\right)\right) = \frac{\partial \omega_j}{\partial v^i} - \frac{\partial \omega_i}{\partial v^j},$$

and hence

$$\frac{\partial \omega_i}{\partial v^j} = \frac{\partial \omega_j}{\partial v^i} \quad \forall 1 \le i, j \le m. \tag{7.7}$$

Then we define (the construction is in analogy with [Lee13, theorem 11.49])

$$u_1(v,w) = \int_\gamma \omega = \int_0^1 \omega(\gamma'(t))\,dt = \int_0^1 \omega_k(tv,w)\,v^k\,dt$$

where $\gamma(t) = (tv, w)$. Clearly $u_1 \in C^\infty(U)$, and by (7.7) we know

$$\begin{aligned} \frac{\partial}{\partial v^i}\left(\omega_k(tv,w)\,v^k\right) &= t\frac{\partial \omega_k}{\partial v^i}(tv,w)\,v^k + \omega_i(tv,w) \\ &= t\frac{\partial \omega_i}{\partial v^k}(tv,w)\,v^k + \omega_i(tv,w) \\ &= \frac{d}{dt}(t\omega_i(tv,w)). \end{aligned}$$

Setting $h_j(v,w) = \int_0^1 \frac{\partial}{\partial w^j}\left(\omega_k(tv,w)\,v^k\right)dt$, it follows that

$$\begin{aligned} du_1(v,w) &= \left(\int_0^1 \frac{d}{dt}(t\omega_i(tv,w))\,dt\right)dv^i + h_j(v,w)dw^j \\ &= \omega_i(v,w)dv^i + h_j(v,w)dw^j = \omega(v,w) + h_j(v,w)dw^j \end{aligned}$$

and hence

$$X_i(u_1) = du_1(X_i) = \omega(X_i) = f_i \quad \forall 1 \le i \le m.$$

Therefore, we find a desired local solution.

For the Cauchy problem (7.5), by [Lee13, corollary 19.13],[22] WLOG we assume that $S \cap U$ is the slice where $v^1 = \cdots = v^m = 0$. Then setting

$$u_0(v,w) = \varphi(0,w) \quad \text{and} \quad u = u_0 + u_1,$$

we get the solution to the Cauchy problem (7.5).

Let $\tilde{u}$ be any other solution to the Cauchy problem (7.5), then $X_i(\tilde{u} - u) = 0$ for each $1 \le i \le m$, and hence $\psi := \tilde{u} - u$ is independent of v. Therefore, $\psi(v,w) = \psi(0,w) = \varphi(0,w) - \varphi(0,w) = 0$, and hence $u = \tilde{u}$. □

More generally, we apply Frobenius theorem to first order overdetermined quasi-linear PDEs.

Proposition 7.12. *Let W be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let $\alpha = (\alpha^i_j) : W \to M(m \times n, \mathbb{R})$ be a smooth matrix-valued function. If the following compatibility conditions*

$$\frac{\partial \alpha^i_j}{\partial x^k} + \alpha^l_k \frac{\partial \alpha^i_j}{\partial z^l} = \frac{\partial \alpha^i_k}{\partial x^j} + \alpha^l_j \frac{\partial \alpha^i_k}{\partial z^l} \quad \forall i,j,k \tag{7.8}$$

hold,[23] *then for any $(x_0, z_0) \in W$, there is a neighborhood U of x_0 in $\mathbb{R}^n$ such that the following overdetermined PDEs (with initial condition)*

$$\begin{cases} \dfrac{\partial u^i}{\partial x^j}(x) = \alpha^i_j\left(x, u^1(x), \cdots, u^m(x)\right), & \forall i,j \\ u(x_0) = z_0 \end{cases} \tag{7.9}$$

admit a unique local solution $u \in C^\infty(U, \mathbb{R}^m)$.

---

[22] This is a nontrivial corollary of [Lee13, theorem 9.46] and Frobenius theorem 7.7.

[23] where we denote a point in $\mathbb{R}^n \times \mathbb{R}^m$ by $(x,z) = (x^1, \cdots, x^n, z^1, \cdots, z^m)$

Proof. The idea is as follows:

(1) Finding u is equivalent to finding $\Gamma(u) = \{(x, u(x)) : x \in U\}$;
(2) Setting $F(x) = (x, u(x))$. The overdetermined PDEs are equivalent to that

$$dF\left(\frac{\partial}{\partial x^i}\Big|_x\right) = \frac{\partial}{\partial x^i}\Big|_{(x,u(x))} + \alpha_i^j(x,u(x))\frac{\partial}{\partial z^j}\Big|_{(x,u(x))} \quad \forall 1 \le i \le n \tag{7.10}$$

which shows the **geometric meaning** of the quasi-linear first order PDEs: the tangent space of $\mathrm{im}(\Gamma)$ is spanned by

$$X_i\Big|_{(x,z)} = \frac{\partial}{\partial x^i}\Big|_{(x,z)} + \alpha_i^j(x,z)\frac{\partial}{\partial z^j}\Big|_{(x,z)}, \quad \text{on } W.$$

(3) To show (7.10), it suffices to show that the distribution D spanned by $X_1, \cdots, X_n$ are involutive, which is guaranteed by the compatibility conditions. Therefore, we can find $\Gamma(u)$ by finding the integral manifold of D.

Define X_i and D as above. Note that D is involutive since we have

$$\begin{aligned}
[X_i, X_j] &= \left[\frac{\partial}{\partial x^i} + \alpha_i^l\frac{\partial}{\partial z^l}, \frac{\partial}{\partial x^j} + \alpha_j^s\frac{\partial}{\partial z^s}\right] \\
&= \frac{\partial \alpha_j^s}{\partial x^i}\frac{\partial}{\partial z^s} + \alpha_i^l\frac{\partial \alpha_j^s}{\partial z^l}\frac{\partial}{\partial z^s} - \frac{\partial \alpha_i^l}{\partial x^j}\frac{\partial}{\partial z^l} - \alpha_j^s\frac{\partial \alpha_i^l}{\partial z^s}\frac{\partial}{\partial z^l} \\
&= \left(\frac{\partial \alpha_j^l}{\partial x^i} + \alpha_i^s\frac{\partial \alpha_j^l}{\partial z^s} - \frac{\partial \alpha_i^l}{\partial x^j} - \alpha_j^s\frac{\partial \alpha_i^l}{\partial z^s}\right)\frac{\partial}{\partial z^l} = 0.
\end{aligned}$$

So given any point $p = (x, z) \in W$, there is an integral manifold N of D containing p. More precisely, by Frobenius theorem 7.7, we suppose that there is a flat chart $(V, (v^1, \cdots, v^n, w^1, \cdots, w^m))$ centered at p such that $N = \Phi^{-1}(0)$ where $\Phi = (w^1, \cdots, w^m)$.

To show the existence of u, by the implicit function theorem [Mei10, theorem 12.5.2] and what we said in the idea, it suffices to show that $\left(\frac{\partial w^i}{\partial z^j}\right)_{m\times m}$ is of rank m. For the sake of convenience, we set

$$P : V \to M(m \times m, \mathbb{R}) \quad \text{where} \quad P_{ij} = \frac{\partial w^i}{\partial z^j},$$

$$Q : V \to M(m \times n, \mathbb{R}) \quad \text{where} \quad Q_{ij} = \frac{\partial w^i}{\partial x^j}.$$

Then we know

$$\mathrm{rank}\left(\frac{\partial w^i}{\partial z^j}\right) = \mathrm{rank}(P) \quad \text{and} \quad \mathrm{rank}([Q, P]) = \mathrm{rank}\left(dw^1, \cdots, dw^m\right) = m.$$

Note that

$$\Phi|_N \equiv 0 \quad \text{and} \quad TN = D|_N \implies d\omega^i(X_j) = 0, \quad \forall i, j.$$

Then we know

$$\frac{\partial \omega^i}{\partial x^j} = -\alpha_j^s\frac{\partial \omega^i}{\partial z^s}, \quad \text{that is,} \quad Q = -P \cdot \alpha.$$

Therefore

$$[Q, P] = [-P \cdot \alpha, P] = P \cdot [-\alpha, I].$$

Hence we know $\operatorname{rank}(P) = \operatorname{rank}([Q, P]) = m$. It follows that $\left(\frac{\partial w^i}{\partial z^j}\right)_{m\times m}$ is of rank m, and hence we prove the existence.

The uniqueness follows immediately from the local structure of integral manifolds [Lee13, proposition 19.16]. □

Corollary 7.13. *Let (M, g) be a (pseudo-)Riemannian manifold, and let $A : T^*M \to \otimes^2 T^*M$ be a smooth map satisfying the following compatibility condition: [where $A(\omega) \in \Gamma(M, \otimes^2 T^*M)$ for all $\omega \in \Gamma(M, T^*M)$]*

$$A(\omega)_{ij;k} - A(\omega)_{ik;j} = R_{jki}{}^{l}\omega_l, \quad \forall \omega \in \Gamma(M, T^*M). \tag{7.11}$$

Then foy any $p \in M$ and every covector $\eta_0 \in T_p^*M$, the overdetermined system of equations

$$\omega_{i;j} = A(\omega)_{ij} \quad \text{i.e.} \quad \nabla\omega(X, Y) = A(\omega)(Y, X) \tag{7.12}$$

admit a smooth solution on a neighborhood of p with $\omega_p = \eta_0$.

Remark 7.14. The following is a **wrong** edition:

"Foy any $p \in M$, there exists a neighborhood U of p such that for any covector $\eta_0 \in T_p^*M$, the overdetermined system of equations $\nabla\omega = A(\omega)$ admit a smooth solution on U with $\omega_p = \eta_0$."

Remark 7.15. If U is a sufficiently small neighborhood U of p (such that the local solution exists), then the solution on U is also unique.

Proof. For $p \in M$, let (x^i) be local coordinates on U centered at p. Then (7.12) becomes

$$\frac{\partial \omega_i}{\partial x^j} = \Gamma_{ij}^s \omega_s + A(\omega)_{ij}$$

where we use the fact that

$$\Gamma_{ij}^k = \Gamma_{ji}^k \quad \forall i, j, k. \tag{7.13}$$

Moreover, it's equivalent to

$$\frac{\partial \omega_i}{\partial x^j}(x) = \alpha_j^i\left(x, \omega_1(x), \cdots, \omega_n(x)\right)$$

where

$$\alpha_j^i\left(x, (z_1, \cdots, z_n)\right) = \Gamma_{ij}^s(x) z_s + A\left(z_s dx^s\right)_{ij}.$$

By proposition 7.12, it suffices to show

$$\frac{\partial \alpha_j^i}{\partial x^k} + \alpha_k^l \frac{\partial \alpha_j^i}{\partial z^l} = \frac{\partial \alpha_k^i}{\partial x^j} + \alpha_j^l \frac{\partial \alpha_k^i}{\partial z^l} \quad \forall i, j, k \quad \text{on } U \times \mathbb{R}^n. \tag{7.14}$$

The idea to verify it is as follows:

(1) If we compute terms like $\partial\alpha_j^i/\partial z^l$ directly, it will be hard to connect the results to our conditions.
(2) In fact, α_j^i is easy to analyze only when we consider $\alpha_j^i\left(x, \omega_1(x), \cdots, \omega_n(x)\right)$, since

$$\alpha_j^i\left(x, \omega_1(x), \cdots, \omega_n(x)\right) = \Gamma_{ij}^s(x)\omega_s(x) + A(\omega)_{ij}(x).$$

(3) The derivative of $\alpha^i_j(x,\omega_1(x),\cdots,\omega_n(x))$ with respect to x^k will also produce the terms like $\partial\alpha^i_j/\partial z^l$:

$$\frac{\partial}{\partial x^k}\Big|_x\left(\alpha^i_j(x,\omega_1(x),\cdots,\omega_n(x))\right)=\frac{\partial\alpha^i_j}{\partial x^k}\Big|_{(x,\omega_1(x),\cdots,\omega_n(x))}+\frac{\partial\alpha^i_j}{\partial z^l}\Big|_{(x,\omega_1(x),\cdots,\omega_n(x))}\frac{\partial\omega_l}{\partial x^k}\Big|_x.$$

(4) Any point $(x,z_1,\cdots,z_n)\in U\times\mathbb{R}^n$ can be expressed as $(x,\omega_1(x),\cdots,\omega_n(x))$ for some $\omega\in\Gamma(U,T^*U)$. Moreover, for a fixed point $(x,z_1,\cdots,z_n)\in U\times\mathbb{R}^n$, we can choose $\omega\in\Gamma(U,T^*M)$ with

$$\omega_k(x)=z_k\quad\forall k\quad\text{and}\quad\frac{\partial\omega_i}{\partial x^j}(x)=\alpha^i_j(x,z)\quad\forall i,j. \tag{7.15}$$

It follows from the idea that

$$\left(\frac{\partial\alpha^i_j}{\partial x^k}+\alpha^l_k\frac{\partial\alpha^i_j}{\partial z^l}\right)(x,z_1,\cdots,z_n)=\frac{\partial}{\partial x^k}\Big|_x\left(\Gamma^s_{ij}\omega_s+A(\omega)_{ij}\right),$$

$$\frac{\partial\omega_s}{\partial x^k}(x)=\alpha^s_k(x,\omega_1(x),\cdots,\omega_n(x))=\Gamma^m_{sk}(x)\omega_m(x)+A(\omega)_{sk}(x).$$

Therefore,

$$\left(\frac{\partial\alpha^j_i}{\partial x^k}+\alpha^l_k\frac{\partial\alpha^j_i}{\partial z^l}\right)(x,z_1,\cdots,z_n)=\left(\left(\partial_k\Gamma^s_{ij}+\Gamma^m_{ij}\Gamma^s_{mk}\right)\omega_s+\Gamma^s_{ij}A(\omega)_{sk}+\partial_kA(\omega)_{ij}\right)(x)$$

and hence using (7.13) we have

$$\begin{aligned}&\left(\frac{\partial\alpha^i_j}{\partial x^k}+\alpha^l_k\frac{\partial\alpha^i_j}{\partial z^l}\right)(x,z_1,\cdots,z_n)-\left(\frac{\partial\alpha^i_k}{\partial x^j}+\alpha^l_j\frac{\partial\alpha^i_k}{\partial z^l}\right)(x,z_1,\cdots,z_n)\\
=&\left(\partial_k\Gamma^s_{ij}+\Gamma^m_{ij}\Gamma^s_{mk}-\partial_j\Gamma^s_{ik}-\Gamma^m_{ik}\Gamma^s_{mj}\right)\omega_s\Big|_x\\
&+\left(\Gamma^s_{ij}A(\omega)_{sk}+\partial_kA(\omega)_{ij}-\Gamma^s_{ik}A(\omega)_{sj}-\partial_jA(\omega)_{ik}\right)\Big|_x\\
=&R_{kji}{}^s\omega_s\Big|_x+\left(A(\omega)_{ij;k}-A(\omega)_{ik;j}\right)\Big|_x=R_{kji}{}^s\omega_s\Big|_x+R_{jki}{}^s\omega_s\Big|_x=0.\end{aligned}$$

We are done. □

Via the proof, we know that although our condition is natural, it's still too strong. Specifically, to show the conclusion, it suffices to show that for any point $(x,z_1,\cdots,z_n)\in U\times\mathbb{R}^n$, there exists $\omega\in\Gamma(U,T^*U)$ such that

$$A(\omega)_{ij;k}\Big|_x-A(\omega)_{ik;j}\Big|_x=R_{jki}{}^l\omega_l\Big|_x \tag{7.16}$$

and that

$$\omega_k(x)=z_k\quad\forall k\quad\text{and}\quad\frac{\partial\omega_i}{\partial x^j}(x)=\Gamma^m_{ij}(x)\omega_m(x)+A(\omega)_{ij}(x)\quad\forall i,j. \tag{7.17}$$

Clearly, there exists $\omega\in\Gamma(U,T^*U)$ satisfying (7.17). It suffices to show that any $\omega\in\Gamma(U,T^*U)$ satisfying (7.17) also satisfies (7.16). Therefore, we have the following conclusion:

Proposition 7.16. *Let (M,g) be a (pseudo-)Riemannian manifold, and let $A:T^*M\to\otimes^2T^*M$ be a smooth map such that $A(\omega)\in\Gamma(M,\otimes^2T^*M)$ for all $\omega\in\Gamma(M,T^*M)$. Suppose* *that for any $x \in M$ and for any $\omega \in \Gamma(M, T^*M)$ with*

$$\omega_{i;j}(x) = A(\omega)_{ij}(x)$$

it holds that

$$A(\omega)_{ij;k}\big|_x - A(\omega)_{ik;j}\big|_x = R_{jki}{}^l \omega_l\big|_x.$$

Then foy any $p \in M$ and every covector $\eta_0 \in T_p^*M$, the overdetermined system of equations

$$\omega_{i;j} = A(\omega)_{ij}$$

admit a smooth solution on a neighborhood of p with $\omega_p = \eta_0$.

Proof. We keep the notations in the proof of corollary 7.13. By conditions it is clear that for any $(x, z_1, \cdots, z_n) \in U \times \mathbb{R}^n$, there exists $\omega \in \Gamma(U, T^*U)$ satisfying (7.16) and (7.17). Then by the proof of corollary 7.13 again, we get the conclusion. □

Example 7.17. Let P be the Schouten tensor, W the Weyl tensor, and C the Cotton tensor (see definitions 1.11, 1.24). Let $A : T^*M \to \otimes^2 T^*M$ be a smooth map given by

$$A(\omega) = \frac{P}{2} + \omega \otimes \omega - \frac{1}{2}\langle \omega, \omega \rangle \cdot g \quad \forall \omega \in \Gamma(M, T^*M).$$

If $W = C = 0$, then foy any $p \in M$ and every covector $\eta_0 \in T_p^*M$, the overdetermined system of equations

$$\omega_{i;j} = A(\omega)_{ij}$$

admit a smooth solution on a neighborhood of p with $\omega_p = \eta_0$.

Proof. By proposition 7.16, it suffices to show that for any $x \in M$ and for any $\omega \in \Gamma(M, T^*M)$ with

$$\omega_{i;j}(x) = A(\omega)_{ij}(x) \tag{7.18}$$

it holds that

$$A(\omega)_{ij;k}\big|_x - A(\omega)_{ik;j}\big|_x = R_{jki}{}^l \omega_l\big|_x.$$

Note that

$$A(\omega)_{ij} = \frac{P_{ij}}{2} + \omega_i \omega_j - \frac{1}{2}\omega^m \omega_m g_{ij}.$$

It follows that

$$A(\omega)_{ij;k} = \frac{P_{ij;k}}{2} + \omega_{i;k}\omega_j + \omega_i \omega_{j;k} - \omega^m \omega_{m;k} g_{ij}$$

and hence

$$\begin{aligned} A(\omega)_{ij;k} - A(\omega)_{ik;j} &= C_{ijk} + \omega_{i;k}\omega_j + \omega_i \omega_{j;k} - \omega^m \omega_{m;k} g_{ij} \\ &\quad - \omega_{i;j}\omega_k - \omega_i \omega_{k;j} + \omega^m \omega_{m;j} g_{ik}. \end{aligned}$$

By formula (7.18), we know

$$
\begin{aligned}
A(\omega)_{ij;k}\Big|_x - A(\omega)_{ik;j}\Big|_x &= C_{ijk}(x) + \left(\frac{P_{ik}}{2} + \omega_i\omega_k - \frac{1}{2}\omega^s\omega_s g_{ik}\right)\omega_j\Big|_x \\
&\quad - \left(\frac{P_{mk}}{2} + \omega_m\omega_k - \frac{1}{2}\omega^s\omega_s g_{mk}\right)\omega^m g_{ij}\Big|_x \\
&\quad - \left(\frac{P_{ij}}{2} + \omega_i\omega_j - \frac{1}{2}\omega^s\omega_s g_{ij}\right)\omega_k\Big|_x \\
&\quad + \left(\frac{P_{mj}}{2} + \omega_m\omega_j - \frac{1}{2}\omega^s\omega_s g_{mj}\right)\omega^m g_{ik}\Big|_x \\
&= \left(C_{ijk} - (P \owedge g)_{mijk}\omega^m\right)\Big|_x .
\end{aligned}
$$

Since $W = C = 0$, it follows that

$$
A(\omega)_{ij;k}\Big|_x - A(\omega)_{ik;j}\Big|_x = -R_{mijk}\omega^m\Big|_x = R_{jki}{}^l\omega_l\Big|_x .
$$

We are done. □

Remark 7.18. It's easy to see that $d\omega = 0$ for the local solution ω, and hence we actually get a local solution of the following second-order PDEs:

$$
P - 2\mathrm{Hess}\, f + 2df \otimes df - \langle \nabla f, \nabla f\rangle \cdot g = 0.
$$

8. Appendix — functional analysis and real analysis

8.A. Lagrange multiplier on Banach spaces.

Definition 8.1. *Let V and W be normed vector spaces, and $U \subset V$ be an open subset of V. A map $f : U \to W$ is called **Fréchet differentiable** at $x \in U$ if there exists a bounded linear operator $A : V \to W$ such that*

$$\lim_{\|h\|\to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0$$

or equivalently

$$f(x+h) = f(x) + Ah + o(h).$$

*If there exists such an operator A, it is unique, so we wirte $Df(x) = A$ and call it the **Fréchet derivative** of f at x.*

A map f that is Fréchet differentiable at any point of U is said to be C^1 if the function

$$Df : U \to B(V, W), \quad x \mapsto Df(x)$$

is continuous, where $B(V, W)$ denotes the normed vector space of all bounded linear operators form V to W.

Theorem 8.2. *Let X and Y be real Banach spaces, let U be an open subset of X, let $f : U \to \mathbb{R}$ be continuously differentiable function, and let $g : U \to Y$ be another continuously differentiable function. Then*

$$\begin{cases} f(u_0) = \inf\limits_{u\in g^{-1}(0)} f(u) & \text{for some } u_0 \in U \\ Dg(u_0) : X \to Y & \text{is surjective} \end{cases} \implies Df(u_0) = \lambda \circ Dg(u_0) \quad \text{for some } \lambda \in Y^*.$$

Proof. See [Zei85, theorem 43.D]. □

8.B. Sobolev inequalities and Poincaré inequality on domains.

Theorem 8.3 (General Sobolev inequalities on domains)**.** *For $\mathbb{R}^n$, we have the following conclusions:*

(1) (Gagliardo-Nirenberg-Sobolev inequality) Assume $1 \le p < n$. There exists a constants $C = C(p, n)$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C_c^1(\mathbb{R}^n).$$

In particular, since $C_c^1(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, we have

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n) \tag{8.1}$$

which implies the continuous embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$.

(2) (Morrey's inequality) Assume $n < p \le \infty$. Then there exists a constant $C = C(p, n)$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in C^1(\mathbb{R}^n),$$

where

$$\gamma := 1 - \frac{n}{p}.$$

In particular, since $C^1(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, we have

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n), \tag{8.2}$$

which implies the continuous embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\gamma}(\mathbb{R}^n)$.

Let U be a bounded open subset of R^n with a C^1 boundary.

(1) If

$$k < \frac{n}{p} \quad and \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

then we have a continuously embedding

$$W^{k,p}(U) \hookrightarrow L^q(U).$$

Specifically, we have the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}} \quad \forall u \in W^{k,p}(U)$$

where $C = C(k, p, n, U)$ is a constant.

(2) If

$$k > \frac{n}{p} \quad and \quad \gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & if\ \frac{n}{p}\ is\ not\ an\ integer \\ any\ positive\ number < 1, & if\ \frac{n}{p}\ is\ an\ integer \end{cases}$$

then we have a continuously embedding

$$W^{k,p}(U) \hookrightarrow C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U}).$$

Specifically, we have the estimate

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})} \leq C\|u\|_{W^{k,p}(U)} \quad \forall u \in W^{k,p}(U)$$

where $C = C(k, p, n, \gamma, U)$ is a constant.

Proof. See [Eva10, section 5.6 — theorems 1, 4, 6]. □

Remark 8.4. In particular, for $n \geq 3$, Gagliardo-Nirenberg-Sobolev inequality (8.1) implies the Sobolev inequality

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq \sigma_n\|Du\|_{L^2(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n) \tag{8.3}$$

where the smallest such constant σ_n is called the n-dimensional **Sobolev constant**.

Theorem 8.5 (Poincaré inequality)**.** *Assume U is a bounded open subset of $\mathbb{R}^n$. Suppose $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)} \quad \forall u \in W_0^{1,p}(U)$$

for each $q \in [1, p^*]$, where $C = C(p, q, n, U)$ is a constant.

Proof. See [Eva10, section 5.6 — theorem 3]. □

Theorem 8.6 (Rellich-Kondrachov compactness theorem)**.** *Assume U us a bounded subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $1 \leq p < n$. Then we have the compact embedding*

$$W^{1,p}(U) \hookrightarrow L^q(U)$$

for each $1 \le q < p^*$.

Proof. See [Eva10, section 5.7 — theorem 1]. □

Remark 8.7. Observe that since $p^* > p$ and $p^* \to \infty$ as $p \to n$, in particular we have the compact embedding

$$W^{1,p}(U) \hookrightarrow L^p(U)$$

for all $1 \le p \le \infty$. Note also that we have the compact embedding

$$W_0^{1,p}(U) \hookrightarrow L^p(U)$$

even if we do not assume ∂U to be C^1.

Theorem 8.8 (Poincaré-Wirtinger inequality). *Let Ω be a domain in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$. Assume $1 \le p \le \infty$. Then there exists a constant $C = C(n, p, \Omega)$ such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C \|Du\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega)$$

where $u_\Omega = (\int_\Omega u\, dx)/(\int_\Omega dx)$.

Proof. See [Eva10, section 5.8 — theorem 1]. □

8.C. **Some basic real analysis.**

Lemma 8.9. *Let Ω be a domain in $\mathbb{R}^n$. Assume $1 \le p < \infty$. If $u_j \to u$ pointwisely, and if $|u_j| \le g$ for some $g \in L^p(\Omega)$, then $u_j \to u$ in $L^p(\Omega)$. In particular, on a compact manifold M, if each u_j is bounded, and if $u_j \to u$ pointwisely, then $u_j \to u$ in $L^p(M)$.*

Proof. It follows from Lebesgue's dominated convergence theorem. □

9. Appendix — second order PDE

Throughout this section we shall denote by $Lu = f$ tbe equation

$$Lu = a^{ij}D_{ij}u + b^i(x)D_i(u) + c(x)u = f(x),$$

where the coefficients and f are defined in an open set $\Omega \subset \mathbb{R}^n$ and, unless otherwise stated, the operator L is *strictly elliptic*; that is,

$$a^{ij}\xi_i\xi_j \geq \lambda|\lambda|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

for some positive constant λ.

9.A. Introduction.

Roughly speaking, we consider the problem

$$L : \mathcal{B} \to \mathcal{V}$$

where we have some basic cases:

$$\mathcal{B} = W^{2,p}(\Omega) \quad \text{and} \quad \mathcal{V} = L^p(\Omega), \tag{9.1}$$

$$\mathcal{B} = C^{2,\alpha}(\Omega) \quad \text{and} \quad \mathcal{V} = C^\alpha(\Omega), \tag{9.2}$$

$$\mathcal{B} = H^1(\Omega) \quad \text{and} \quad \mathcal{V} = H^{-1}(\Omega). \tag{9.3}$$

First, we put forward a basic method to deal with cases (9.1) and (9.2).

(1) Using **priori estimates** and **continuity method**, we show that $L(\mathcal{B}) = \mathcal{V}$ is equivalent to $\Delta(\mathcal{B}) = \mathcal{V}$. [$L(\mathcal{B}) = \mathcal{V}$ means that the equation always has a solution.[24]]
(2) By **Perron process** and **priori estimates**, $\Delta(\mathcal{B}) = \mathcal{V}$ follows.
Specifically, Perron process helps us find the solution, and priori estimates shows that the solution has some regularity.

Second, we put forward a basic method to deal with case (9.3).

(1) Setting $a(u, v) = (Lu, v)$, then by **Lax-Milgram theorem** [Xio, corollary 4.20], the Dirichlet problem has a (unique) solution if a is *coercive*.
(2) Via integration by parts, we can write down a exactly. Then by **Poincaré inequality** and basic estimates, we know $L + \mu : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism for sufficiently large μ, since the corresponding bounded bilinear form is coercive.

Note that up to now we have considered sufficient conditions for $L(\mathcal{B}) = \mathcal{V}$. Generally, this doesn't hold, but elliptic operator L still has the **alternative** property:

either one can always solve $Lx = y$,
or else $0 < \dim\ker L^* < \infty$, in which case a solution exists iff $y \perp \ker L^*$.

Both the above two methods can show the alternative property:

(1) For cases (9.1) and (9.2), the **priori estimates** show that

$L(\mathcal{B})$ is closed and $\ker L$ is fintie dimensional.

Then the alternative conclusion follows from [Xio, theorem 3.15].
(2) For case (9.3), by compact embedding we know $(L + \mu)^{-1}$ is **compact**, and hence the conclusion follows from **Fredholm alternative**.

[24]By the **maximum principle**, L is a *positive operator*, and hence the solution must be unique.

9.B. **Continuity method.**

Theorem 9.1 (Continuity method)**.** *Let $\mathcal{B}$ be a Banach space, let $\mathcal{V}$ be a normed vector space, and let $L_0, L_1 : \mathcal{B} \to \mathcal{V}$ be bounded linear operators. For each $t \in [0,1]$, set*

$$L_t = (1-t)L_0 + tL_1$$

and suppose that there is a constant C such that

$$\|x\|_{\mathcal{B}} \le C \|L_t x\|_{\mathcal{V}}, \quad \forall x \in \mathcal{B}, \quad \forall t \in [0,1] \tag{9.4}$$

Then $\operatorname{im}(L_1) = \mathcal{V}$ *iff* $\operatorname{im}(L_2) = \mathcal{V}$.

Proof. Condition (9.4) shows that L_t is injective for each $t \in [0,1]$. Suppose that L_s is onto for some $s \in [0,1]$. Therefore, L_s is bijective. Then given any $t \in [0,1]$ and given any $y \in \mathcal{V}$, we have

$$\begin{aligned} L_t x = y &\iff ((1-t)L_0 + tL_1)(x) = y \\ &\iff ((1-s)L_0 + sL_1)(x) + (s-t)(L_0 - L_1)(x) = y \\ &\iff L_s(x) + (s-t)(L_0 - L_1)(x) = y \\ &\iff x = L_s^{-1}(y) - (s-t)L_s^{-1}(L_0 - L_1)(x). \end{aligned}$$

To show such x exists (for $L_t x = y$), by Banach fixed point theorem [Xio, theorem 2.14] it suffices to show that

$$T : \mathcal{B} \to \mathcal{B}, \quad x \mapsto L_s^{-1}(y) - (s-t)L_s^{-1}(L_0 - L_1)(x)$$

is a contraction mapping. By condition (9.4), clearly, T is a contraction map if

$$|s-t| < \frac{1}{C\left(\|L_0\| + \|L_1\|\right)}$$

By compactness of $[0,1]$, the conclusion follows. □

9.C. **Schauder estimate on domains, solving PDE on Hölder spaces.**

Remark 9.2. A $C^{k,\alpha}$ $(0 \le \alpha \le 1)$ domain is already bounded. One can refer to [GT01, section 6.2] for its detailed definition.

Theorem 9.3 (Schauder estimate)**.** *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$. Suppose that*

$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n,$$
$$\left|a^{ij}\right|_{0,\alpha}, \left|b^i\right|_{0,\alpha}, |c|_{0,\alpha} \le \Lambda$$

for some positive constants λ and Λ. If

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = \phi & \text{on} \quad \partial\Omega \end{cases} \qquad \text{where} \quad u \in C^{2,\alpha}(\overline{\Omega}),\ f \in C^{\alpha}(\overline{\Omega}),\ \phi \in C^{2,\alpha}(\overline{\Omega})$$

then

$$|u|_{2,\alpha;\Omega} \le C\left(|u|_{0;\Omega} + |\phi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}\right) \tag{9.5}$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$ is a constant.

Proof. See [GT01, theorem 6.6]. □

Theorem 9.4. *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$, and let L be strictly epplitic in Ω with coefficients in $C^\alpha(\overline{\Omega})$ and with $c \leq 0$. Then if the Dirichlet problem for Poisson's equation*

$$\begin{cases} \Delta u = f & in \quad \Omega \\ u = \phi & on \quad \partial\Omega \end{cases}$$

has a $C^{2,\alpha}(\overline{\Omega})$ solution for all $f \in C^\alpha(\overline{\Omega})$ and all $\phi \in C^{2,\alpha}(\overline{\Omega})$, the problem

$$\begin{cases} Lu = f & in \quad \Omega \\ u = \phi & on \quad \partial\Omega \end{cases}$$

also has a (unique) $C^{2,\alpha}(\overline{\Omega})$ solution for all $f \in C^\alpha(\overline{\Omega})$ and all $\phi \in C^{2,\alpha}(\overline{\Omega})$.

Proof. By setting $v = u - \phi$, it suffices to restrict consideration to zero boundary values.

In the next we apply the continuity method 9.1. For $t \in [0,1]$ we set

$$L_t = tL + (1-t)\Delta : \mathcal{B}_1 \to \mathcal{B}_2$$

where Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\mathcal{B}_1 = \left\{ u \in C^{2,\alpha}(\overline{\Omega}) : u = 0 \text{ on } \Omega \right\} \quad \text{and} \quad \mathcal{B}_2 = C^\alpha(\overline{\Omega}).$$

By hypothesis we may assume that the coefficients of L_t satisfies

$$\begin{aligned} &\lambda|\xi|^2 \leq a^{ij}(t)\xi_i\xi_j \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n,\ t \in [0,1], \\ &\left|a^{ij}(t)\right|_{0,\alpha}, \left|b^i(t)\right|_{0,\alpha}, |c(t)|_{0,\alpha} \leq \Lambda \quad \forall t \in [0,1], \end{aligned} \tag{9.6}$$

for some positive constants λ and Λ. Clearly by (9.6), we know $L_t : \mathcal{B}_1 \to \mathcal{B}_2$ is well-defined and bounded for each $t \in [0,1]$.

By the maximum-estimate theorem [GT01, theorem 3.7], we know furthermore that

$$|u|_0 \leq C_1 \sup_\Omega |L_t u| \leq C_1 |L_t u|_{0,\alpha} \quad \forall t \in [0,1]$$

where $C_1 = C_1(\lambda, \Lambda, \mathrm{diam}(\Omega))$ is a constant. By Schauder estimate (9.5) we know

$$|u|_{2,\alpha} \leq C|L_t u|_{0,\alpha} \quad \forall t \in [0,1]$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$ is a new constant. That is,

$$||u||_{\mathcal{B}_1} \leq C\, ||L_t u||_{\mathcal{B}_2} \quad \forall t \in [0,1].$$

Then the conclusion follows from the continuity method 9.1. □

Then by the conclusion for the model case $L = \Delta$, we have the following conclusion:

Theorem 9.5. *Let Ω be a $C^{2,\alpha}$ domain, and let L be strictly epplitic in Ω with coefficients in $C^\alpha(\overline{\Omega})$ and with $c \leq 0$. Then the Dirichlet problem,*

$$\begin{cases} Lu = f & in \quad \Omega \\ u = \phi & on \quad \partial\Omega \end{cases} \quad \textit{where} \quad f \in C^\alpha(\overline{\Omega}), \quad \phi \in C^{2,\alpha}(\overline{\Omega})$$

has a (unique) solution lying in $C^{2,\alpha}(\overline{\Omega})$.

Proof. It follows from theorem 9.4 and subsequent theorem 9.11. □

9.D. L^p **estimate on domains, solving PDE on Sobolev spaces.**

Remark 9.6. A $C^{k,\alpha}$ $(0 \le \alpha \le 1)$ domain is already bounded. One can refer to [GT01, section 6.2] for its detailed definition.

For the sake of convenience, given a domain Ω and a function $f : \Omega \to \mathbb{R}$, we denote the moduli of continuity of f by

$$|f|_{mc;\Omega} = \inf\{\mu : |f(x) - f(y)| \le \mu|x-y|,\ \forall x, y \in \Omega\}.$$

Theorem 9.7 (L^p estimate)**.** *Let Ω be a domain in $\mathbb{R}^n$ with a $C^{1,1}$ boundary portion $T \subset \partial\Omega$. Suppose that*

$$\begin{aligned} &a^{ij} \in C^0(\Omega \cup T), \quad b^i, c \in L^\infty(\Omega), \\ &\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \\ &|a^{ij}|, |b^i|, |c| \le \Lambda \end{aligned}$$

for some positive constants λ and Λ. If $1 < p < \infty$ and if

$$\begin{cases} Lu = f & \text{in} \quad \Omega \quad \text{[strong solution]} \\ u = 0 & \text{on} \quad T \quad \text{[in the sense of } W^{1,p}(\Omega)] \end{cases} \qquad \text{where} \quad u \in W^{2,p}(\Omega),\ f \in L^p(\Omega)$$

then for any domain $\Omega' \subset\subset \Omega \cup T$, we have

$$\|u\|_{2,p;\Omega'} \le C\left(\|u\|_{p;\Omega} + \|f\|_{p;\Omega}\right) \tag{9.7}$$

where $C = C(n, p, \lambda, \Lambda, \Omega', \Omega, |a^{ij}|_{mc;\Omega'})$ is a constant.

Proof. See [GT01, theorem 9.13]. ☐

Corollary 9.8. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$. Suppose that*

$$\begin{aligned} &a^{ij} \in C^0(\overline{\Omega}), \quad b^i, c \in L^\infty(\Omega), \quad c \le 0 \\ &\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \end{aligned}$$

for some positive constant λ. Then for $1 < p < \infty$, we have

$$\|u\|_{2,p;\Omega} \le C\|Lu\|_{p;\Omega} \quad \forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \tag{9.8}$$

where $C = C(n, p, \lambda, \Lambda, \Omega, \mu)$ is a constant, where[25]

$$\mu = \max|a^{ij}|_{mc;\Omega} \quad \text{and} \quad \Lambda = \max\left\{\sup|a^{ij}|, \sup|b^i|, \sup|c|\right\}.$$

Proof. By L^p estimate 9.7, it suffices to show that

$$\|u\|_{p;\Omega} \le C\|Lu\|_{p;\Omega} \quad \forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

for some constant C. Suppose for contradiction that there exists a sequence $(v_m) \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying

$$\|v_m\|_{p;\Omega} = 1, \quad \|Lv_m\|_{p;\Omega} \to 0, \quad \forall m \in \mathbb{N}.$$

Then via L^p estimate 9.7, the weak compactness of bounded sets in $W^{2,p}(\Omega)$, and the compact embedding $W_0^{1,p} \hookrightarrow L^p(\Omega)$, there exists a subsequence, which we relabel as (v_m), converging weakly (both in $W^{2,p}$ and in $W_0^{1,p}$) to a function $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

[25] Since $\overline{\Omega}$ is compact, μ and Λ are finite.

satisfying $||v||_{p;\Omega} = 1$. Since

$$\int_\Omega gD^\alpha v_m \to \int_\Omega gD^\alpha v, \quad \forall|\alpha| \le 2, \quad \forall g \in L^{p/(p-1)}(\Omega),$$

we must have

$$\int_\Omega gLv = 0, \quad \forall g \in L^{p/(p-1)}(\Omega),$$

and hence $Lv = 0$. By uniqueness we know $v = 0$ (using the maximum principle), which contradicts the condition $||v||_p = 1$. □

Theorem 9.9. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$. Given $1 < p < \infty$. Suppose that*

$$a^{ij} \in C^0(\overline{\Omega}), \quad b^i, c \in L^\infty(\Omega), \quad c \le 0$$
$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constant λ. If the Dirichlet problem for Poisson's equation

$$\begin{cases} \Delta u = f & \text{in} \quad \Omega \quad \text{[strong solution]} \\ u - \phi \in W_0^{1,p} \end{cases}$$

has a $W^{2,p}(\Omega)$ solution for all $f \in L^p(\Omega)$ and all $\phi \in W^{2,p}(\Omega)$, the problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \quad \text{[strong solution]} \\ u - \phi \in W_0^{1,p} \end{cases}$$

also has a (unique) $W^{2,p}(\Omega)$ solution for all $f \in L^p(\Omega)$ and all $\phi \in W^{2,p}(\Omega)$,.

Proof. By setting $v = u - \phi$, it suffices to restrict consideration to zero boundary values.

In the next we apply the continuity method 9.1. For $t \in [0, 1]$ we set

$$L_t = tL + (1 - t)\Delta : \mathcal{B}_1 \to \mathcal{B}_2$$

where Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\mathcal{B}_1 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \text{and} \quad \mathcal{B}_2 = L^p(\Omega)$$

where $\mathcal{B}_1$ is equipped with the norm $||\bullet||_{2,p;\Omega}$. By hypothesis we may assume that the coefficients of L_t satisfies

$$\begin{aligned} &a^{ij}(t) \in C^0(\overline{\Omega}), \quad b^i(t), c(t) \in L^\infty(\Omega), \quad \forall t \in [0, 1], \\ &\lambda|\xi|^2 \le a^{ij}(t)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad \forall t \in [0, 1], \\ &|a^{ij}(t)|, |b^i(t)|, |c(t)| \le \Lambda, \quad \forall t \in [0, 1], \\ &|a^{ij}(t)|_{mc;\Omega} \le \mu, \quad c(t) \le 0, \quad \forall t \in [0, 1], \end{aligned} \tag{9.9}$$

for some positive constants λ, Λ and μ. Clearly by (9.9), we know $L_t : \mathcal{B}_1 \to \mathcal{B}_2$ is well-defined and bounded for each $t \in [0, 1]$. By the L^p estimate (9.8) for $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, we know that

$$||u||_{2,p;\Omega} \le C||L_t u||_{p;\Omega} \quad \forall u \in \mathcal{B}_1 \quad \forall t \in [0, 1]$$

where $C = C(n, p, \lambda, \Lambda, \Omega, \mu)$ is a constant. That is,

$$||u||_{\mathcal{B}_1} \le C\, ||L_t u||_{\mathcal{B}_2} \quad \forall u \in \mathcal{B}_1 \quad \forall t \in [0, 1].$$

Then the conclusion follows from the continuity method 9.1. □

Then by the conclusion for the model case $L = \Delta$, we have the following conclusion:

Theorem 9.10. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$. Suppose that*

$$a^{ij} \in C^0(\overline{\Omega}), \quad b^i, c \in L^\infty(\Omega), \quad c \le 0$$
$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constant λ. Then for $1 < p < \infty$, the Dirichlet problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \quad \text{[strong solution]} \\ u - \phi \in W_0^{1,p} \end{cases} \qquad \text{where} \quad f \in L^p(\Omega), \quad \phi \in W^{2,p}(\Omega)$$

has a unique solution $u \in W^{2,p}(\Omega)$.

Proof. It follows from theorem 9.9 and subsequent theorem 9.11. □

9.E. **The model case:** $L = \Delta$**.** For the model case $L = \Delta$, Perron process helps us find the solution, and priori estimates shows that the solution has some regularity.

Theorem 9.11. *Here are two basic conclusions for the existence of Poisson's equations:*
(1) If Ω is a $C^{2,\alpha}$ domain in $\mathbb{R}^n$, then the Dirichlet problem for Poisson's equation

$$\begin{cases} \Delta u = f & \text{in} \quad \Omega \\ u = \phi & \text{on} \quad \partial\Omega \end{cases}$$

has a $C^{2,\alpha}(\overline{\Omega})$ solution for all $f \in C^\alpha(\overline{\Omega})$ and all $\phi \in C^{2,\alpha}(\overline{\Omega})$.
(2) If Ω is a $C^{1,1}$ domain in $\mathbb{R}^n$, then the Dirichlet problem for Poisson's equation

$$\begin{cases} \Delta u = f & \text{in} \quad \Omega \quad \text{[strong solution]} \\ u - \phi \in W_0^{1,p} \end{cases}$$

has a $W^{2,p}(\Omega)$ solution for all $f \in L^p(\Omega)$ and all $\phi \in W^{2,p}(\Omega)$.

Proof. Point (1) follows from [GT01, theorem 6.11, lemma 6.12, remarks in section 6.3].

In the next we prove point (2). By setting $v = u - \phi$, it suffices to restrict consideration to the case $\phi = 0$. Then let $(f_n) \in C_c^\infty(M)$ satisfying $f_n \to f$ in $L^p(\Omega)$, and let (u_n) be the solutions to

$$\begin{cases} \Delta u_n = f_n & \text{in} \quad \Omega \\ u_n = 0 & \text{on} \quad \overline{\Omega} \setminus \operatorname{supp}(f_n). \end{cases}$$

By point (1), it's clear that $u_n \in C_c^\infty(\Omega)$. By L^p estimate (9.8) for $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$,[26] we know (u_n) converges in $W^{2,p}(\Omega)$. Say $u \in W^{2,p}(\Omega)$ with $u_n \to u$ in $W^{2,p}(\Omega)$. Clearly u is the solution as desired. □

9.F. **Lax-Milgram theorem, solving PDE on** $H^1(\Omega)$**.**

Theorem 9.12 (Lax-Milgram theorem)**.** *Let H be a Hilbert space, and let ϕ be a bounded and coercive bilinear form. Then for all $f \in H^*$, there exists a unique $y \in H$ such that*

$$f(x) = \phi(x, y), \quad \forall x \in H.$$

[26] This property uses the condition that Ω is a $C^{1,1}$ domain.

Proof. See [Xio, corollary 4.20]. □

Theorem 9.13. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Given the elliptic operator*

$$L : H^1(\Omega) \to H^{-1}(\Omega), \quad u \mapsto -\partial_j\left(a^{ij}\partial_i u + d^j u\right) + b^i\partial_i u + cu,$$

where $a^{ji} = a^{ij}$, $a^{ij} \in L^\infty(\Omega)$ and there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \quad \forall x \in \Omega, \tag{9.10}$$

$$\sum_{i=1}^n \left\|b^i\right\|_{L^n(\Omega)} + \sum_{i=1}^n \left\|d^i\right\|_{L^n(\Omega)} + \|c\|_{L^{n/2}(\Omega)} \le \Lambda. \tag{9.11}$$

Suppose that $v \in H^{-1}(\Omega)$, $g \in H^1(\Omega)$. Then there exist $\overline{\mu} > 0$, such that for $\mu \ge \overline{\mu}$, the Dirichlet problem

$$\begin{cases} Lu + \mu jiu = v \\ u - g \in H_0^1(\Omega) \end{cases}$$

has a unique solution $u \in H^1(\Omega)$, where

$$i : H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is the compact embedding.}$$
$$j : L^2(\Omega) \to H^{-1}(\Omega), \quad u \mapsto (u, \cdot)_{L^2(\Omega)}.$$

Proof. Note that the Dirichlet problem can be transformed into finding $u \in H_0^1(\Omega)$ such that $Lu + \mu jiu = w$, where $w \in H^{-1}(\Omega)$. Since

$$\begin{aligned} \langle Lu + \mu jiu, v\rangle &= \left\langle -\partial_j\left(a^{ij}\partial_i u + d^j u\right) + b^i\partial_i u + (c+\mu)u, v\right\rangle \\ &= \left\langle a^{ij}\partial_i u + d^j u, \partial_j v\right\rangle + \left\langle b^i\partial_i u + (c+\mu)u, v\right\rangle \end{aligned}$$

the equation is equivalent to finding $u \in H_0^1(\Omega)$ such that $a(u, \cdot) = w$, where

$$\begin{aligned} a : H_0^1(\Omega) \times H_0^1(\Omega) &\to \mathbb{R} \\ (u, v) &\mapsto \int_\Omega \left(a^{ij}\partial_i u\partial_j v + d^j u\partial_j v + b^i(\partial_i u)v + (c+\mu)uv\right) dx \end{aligned}$$

is a continuous bilinear form.[27] Now we claim that

$$(\star) : \quad \text{There exists } \overline{\mu} > 0 \text{ such that } a \text{ is coercive for } \mu \ge \overline{\mu}.$$

Note that the conclusion will follow from $(\star)$ by Lax–Milgram theorem 9.12. Thus it suffices to prove $(\star)$.

Claim $(\star)$ is easy if the coefficients are in $L^\infty(\Omega)$, but we only have (9.11). Our idea is to show that the gap between (9.11) and L^∞ can be controlled.

Note that via Poincaré inequality 8.5 there exists $c_0 > 0$ such that

$$\|\nabla u\|_{L^2(\Omega)} \ge \frac{2c_0}{\lambda}\|u\|_{H_0^1(\Omega)} \quad \forall u \in H_0^1(\Omega).$$

[27] Continuity (i.e. boundedness) follows from (9.10), (9.11), Hölder inequality and

$$|xAy^\top| \le \sqrt{xAy^\top(xAy^\top)^\top} = \sqrt{xA(y^\top y)Ax^\top} = |y|\sqrt{xAAx^\top} = \frac{|y|}{|x|}\sqrt{(xAx^\top)(xAx^\top)} \le \Lambda|x||y|$$

Then we choose $0 < \varepsilon < c_0$ and find b_1^i, b_2^i, d_1^i, d_2^i, c_1, c_2 and k via the subsequent lemma 9.14 such that

$$\begin{aligned}\sum_{i=1}^n \left\|b_1^i\right\|_{L^\infty(\Omega)} + \sum_{i=1}^n \left\|d_1^i\right\|_{L^\infty(\Omega)} + \|c_1\|_{L^\infty(\Omega)} &\leq k, \\ \sum_{i=1}^n \left\|b_2^i\right\|_{L^n(\Omega)} + \sum_{i=1}^n \left\|d_2^i\right\|_{L^n(\Omega)} + \|c_2\|_{L^{n/2}(\Omega)} &\leq \varepsilon.\end{aligned}$$

Set

$$\begin{aligned}a_1(u,v) &= \int_\Omega \left(a^{ij}\partial_i u \partial_j v + d_1^j u \partial_j v + b_1^i(\partial_i u)v + c_1 uv\right) dx, \\ a_2(u,v) &= \int_\Omega \left(d_2^j u \partial_j v + b_2^i(\partial_i u)v + c_2 uv\right) dx, \\ a_3(u,v) &= \left(k + \frac{2k^2}{\lambda}\right)\int_\Omega uv\, dx.\end{aligned}$$

Then by Hölder inequality we know that[28]

$$\begin{aligned}a_1(u,u) &\geq \lambda \|\nabla u\|_{L^2(\Omega)}^2 - k\left(\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\right) \\ &= \lambda \|\nabla u\|_{L^2(\Omega)}^2 - k\|u\|_{L^2(\Omega)}^2 - 2k\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &= \frac{\lambda}{2}\|\nabla u\|_{L^2(\Omega)}^2 - \left(k + \frac{2k^2}{\lambda}\right)\|u\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{2k^2}{\lambda}\|u\|_{L^2(\Omega)}^2 - 2k\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\geq \frac{\lambda}{2}\|\nabla u\|_{L^2(\Omega)}^2 - \left(k + \frac{2k^2}{\lambda}\right)\|u\|_{L^2(\Omega)}^2 \\ &\geq c_0\|u\|_{H_0^1(\Omega)}^2 - \left(k + \frac{2k^2}{\lambda}\right)\|u\|_{L^2(\Omega)}^2\end{aligned}$$

and that

$$|a_2(u,u)| \leq \varepsilon \|u\|_{H_0^1(\Omega)}^2.$$

Therefore,

$$a_1(u,u) + a_2(u,u) + a_3(u,u) \geq (c_0 - \varepsilon)\|u\|_{H_0^1(\Omega)}^2$$

which proves claim (⋆). Hence the conclusion follows. □

Lemma 9.14. *Given $f \in L^p(\Omega)$ and $\varepsilon > 0$. Then we can find $f = f_1 + f_2$ such that*

$$\sup_{x\in\Omega}|f_1(x)| \leq k(\varepsilon) \qquad \|f_2\|_{L^p(\Omega)} \leq \varepsilon$$

Proof. Put $A_k = \{x \in \Omega : |f| < k\}$, $B_k = \Omega \setminus A_k$ and

$$f_{1k} = f\chi_{A_k} \qquad f_{2k} = f\chi_{B_k}$$

Then we know

$$f_{1k} + f_{2k} = f \qquad \sup_{x\in\Omega}|f_1(x)| \leq k$$

[28] If we use $c_1\|\nabla u\|_{L^2(\Omega)}^2 + c_2\|u\|_{L^2(\Omega)}^2$ to controll $-\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$, we can use a small c_1. But if we use $c_1\|\nabla u\|_{L^2(\Omega)}^2$ and Poincaré inequality to controll $-\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$, we can't use an arbitrarily small c_1.

for all k. Note that $f \in L^p(\Omega)$ implies that $m(B_k) \to 0$ as $k \to \infty$, and then via the Lebesgue dominated convergence theorem [Xio, theorem 9.25] we know

$$\lim_{k\to+\infty} \|f_{2k}\|_{L^p(\Omega)} = 0$$

Thus for any $\varepsilon > 0$, we can find an appropriate k such that f_{1k} and f_{2k} satisfy the requirements. □

Corollary 9.15. *In theorem 9.13, if $\mu \geq \overline{\mu}$, the operator $L + \mu ji : H_0^1(\Omega) \to H^{-1}(\Omega)$ is actually an isomorphism.*

Proof. Note that $L + \mu ji : H_0^1(\Omega) \to H^{-1}(\Omega)$ is continuous since the corresponding bilinear form a is continuous. Thus the conclusion follows from theorem 9.13 and theorem [Xio, theorem 3.22]. □

Example 9.16. The operator $-\Delta + cji : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, where $c \geq 0$ a.e. and $c \in L^{n/2}(\Omega)$.

Proof. The corresponding bilinear form is

$$a(u, v) = \int_\Omega (\partial_i u \cdot \partial_i v + cuv)\, dx$$

Then via Hölder inequality we have

$$|a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + c_1 \|u\|^2_{H_0^1(\Omega)} \|v\|^2_{H_0^1(\Omega)} \leq c_2 \|u\|^2_{H_0^1(\Omega)} \|v\|^2_{H_0^1(\Omega)},$$

and via Poincaré inequality we have

$$a(u, u) \geq \|\nabla u\|^2_{L^2(\Omega)} \geq c_3 \|u\|^2_{H_0^1(\Omega)}.$$

Hence the conclusion follows. □

9.G. General cases on $W^{2,p}(\Omega)$.

The idea has been introduced in subsection 9.A.

Lemma 9.17. *Let E and F be Banach, and let $L : E \to F$ be a bounded linear operator Then the following properties are equivalent:*

(1) $\mathrm{im}(L)$ *is closed.*
(2) $\mathrm{im}(L) = \ker(L^*)^\perp$.

Proof. See [Xio, theorem 3.51]. □

Lemma 9.18. *Let X, Y, Z be reflexive Banach spaces with $X \hookrightarrow Y$ a compact embedding, and let $L : X \to Z$ be a continuous linear operator. Then the following properties are equivalent:*

(1) $\mathrm{im}(L)$ *is closed and* $\ker L$ *is finite dimensional.*
(2) There are constants c_1 and c_2 such that

$$\|x\|_X \leq c_1 \|Lx\|_Z + c_2 \|x\|_Y \tag{9.12}$$

Proof. (1) $\Longrightarrow$ (2): Since $\ker L$ is fintie-dimensional, by [Xio, lemma 3.46] there exists a closed linear subspace $X_1 \subset X$ with $X = X_1 \oplus \ker L$, and by [Xio, theorem 3.15] there exist positive constants b_1 and b_2 with

$$b_1 \|v\|_Y \leq \|v\|_X \leq b_2 \|v\|_Y, \quad \forall v \in \ker L.$$

Moreover, since $L|_{X_1}$ is injective and $\mathrm{im}(L|_{X_1}) = \mathrm{im}(L)$ is closed and hence Banach, by [Xio, theorem 3.22], there exists positive constant b_3 with

$$\|v\|_X \le b_3\|Lv\|_Y, \quad \forall v \in X_1.$$

Therefore, for any $x \in X$, we write $x = x_1 + x_2$ in a unique way due to the direct sum $X = X_1 \oplus \ker L$; then we have

$$\begin{aligned} \|x\|_X &\le \|x_1\|_X + \|x_2\|_X \\ &\le b_3\|Lx_1\|_Y + b_2\|x_2\|_Y = b_3\|Lx\|_Y + b_2\|x - x_1\|_Y \\ &\le b_3\|Lx\|_Y + b_2\|x\|_Y + b_2\|x_1\|_Y \\ &\le b_3\|Lx\|_Y + b_2\|x\|_Y + \frac{b_2}{b_1}\|x_1\|_X \\ &\le b_3\|Lx\|_Y + b_2\|x\|_Y + \frac{b_2 b_3}{b_1}\|Lx\|_Y. \end{aligned}$$

(2) $\implies$ (1): Condition (9.12) implies that

$$\|x\|_X \le c_2\|x\|_Y, \quad \forall x \in \ker L.$$

By the compact embedding $X \hookrightarrow Y$, it follows that $\|\bullet\|_X$ and $\|\bullet\|_Y$ are equivalent norms on $\ker L$. Moreover, with $\ker L$ equipped with the norm $\|\bullet\|_X$, the compact embedding $X \hookrightarrow Y$ and the norm equivalence imply that the unit ball in $\ker L$ is sequentially compact. Therefore, by [Xio, corollary 3.19], $\ker L$ is finite-dimensional.

In the next we prove that $\mathrm{im}(L)$ is closed. By [Xio, lemma 3.46] we decompose $X = X_1 \oplus \ker L$ where X_1 is a closed linear subspace. By [Xio, theorem 3.32], we know X_1 is also a reflexive Banach space. Suppose $Lx_i \to z$ for some $(x_i) \subset X_1$.

(1) First we prove that (x_i) is bounded. Suppose not; there exists a subsequence, which we relabel as (x_i), satisfying $\|x_i\|_X \to \infty$. Setting

$$y_i = \frac{x_i}{\|x_i\|_X}, \quad \forall i = 1, 2, \cdots.$$

then (y_i) is a bounded sequence in X_1 and

$$Ly_i \to 0 \quad \text{in} \quad Z.$$

By [Xio, theorem 3.41], there exists a subsequence, which we relabel as (y_i), satisfying $y_i \rightharpoonup y$ in X_1 for some $y \in X_1$. By compact embedding $X \hookrightarrow Y$ and [Xio, proposition 3.55], we know

$$y_i \to y \quad \text{in} \quad Y.$$

Then condition (9.12) implies that $y_i \to y$ in X_1. Therefore, $Ly_i \to Ly$ and hence $Ly = 0$; i.e. $y \in \ker L$. Since $y \in \ker L \cap X_1$, $y = 0$. However $y_i \to y$ also implies $\|y\| = \lim_i \|y_i\| = 1$. A contraction.

(2) Then we show that (x_i) has a convergent subsequence. Since (x_i) is bounded, by [Xio, theorem 3.41], there exists a subsequence, which we relabel as (x_i), satisfying $x_i \rightharpoonup x$ in X_1 for some $x \in X_1$. By compact embedding $X \hookrightarrow Y$, [Xio, proposition 3.55], and condition (9.12), we know $x_i \to x$ in X_1.

Therefore $z = \lim_i Lx_i = Lx$, which implies $\mathrm{im}(L)$ is closed. □

Since $C^{k,\alpha}(\overline{\Omega})$ is not reflexive,[29] we only apply lemmas 9.17 and 9.18 to elliptic PDEs on $W^{2,p}(\Omega)$.

Theorem 9.19 (Solvability of elliptic PDE on $W^{2,p}(\Omega)$)**.** *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$. Given $1 < p < \infty$. Suppose that*

$$a^{ij} \in C^0(\overline{\Omega}), \quad b^i, c \in L^\infty(\Omega)$$
$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constant λ. Then

$$L^p(\Omega) = L\left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right) \oplus \ker L^*$$

Proof. Since we have the compact embedding $W^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$, the conclusion follows from L^p estimate 9.7, lemma 9.17 and lemma 9.18. □

9.H. **General cases on $H^1(\Omega)$.** The idea has been introduced in subsection 9.A.

Theorem 9.20 (Solvability of elliptic PDE on $H^1(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Consider the elliptic operator*

$$L : H^1(\Omega) \to H^{-1}(\Omega), u \mapsto -\partial_j\left(a^{ij}\partial_i u + d^j u\right) + b^i\partial_i u + cu,$$

where $a^{ji} = a^{ij}$, $a^{ij} \in L^\infty(\Omega)$ and there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \quad \forall x \in \Omega;$$
$$\sum_{i=1}^n \left\|b^i\right\|_{L^n(\Omega)} + \sum_{i=1}^n \left\|d^i\right\|_{L^n(\Omega)} + \|c\|_{L^{n/2}(\Omega)} \le \Lambda.$$

Suppose that $v \in H^{-1}(\Omega)$. Then for the Dirichlet problem

$$Lu = v, \quad u \in H_0^1(\Omega),$$

we have

(1) either *for every $v \in H^{-1}(\Omega)$ the equation has a unique solution,*
(2) or *the homogeneous equation $Lu = 0$ admits n linearly independent solutions, and in this case, the inhomogeneous equation $Lu = v$ is solvable iff v satisfies n orthogonal conditions; that is, $v \in N(I - T^*)^\perp$, where $T = \mu(L + \mu)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$ and μ is an appropriate constant with $L + \mu$ being an isomorphism.*

Proof. Note that the equation is equivalent to finding $u \in H_0^1(\Omega)$ such that

$$(L + \mu ji)u - \mu jiu = v,$$

where $\mu \in \mathbb{R}$ and

$$i : H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is the compact embedding;}$$
$$j : L^2(\Omega) \to H^{-1}(\Omega), \quad u \mapsto (u, \cdot)_{L^2(\Omega)}.$$

By corollary 9.15, we find an appropriate μ with $L+\mu ji$ being an isomorphism. Therefore, the elliptic equation is equivalent to finding $u \in H_0^1(\Omega)$ such that

$$u - Tu = w,$$

[29]See https://math.stackexchange.com/questions/388129.

where

$$T = \mu(L + \mu ji)^{-1} ji, \quad \text{and} \quad w = (L + \mu ji)^{-1}(v) \in H_0^1(\Omega).$$

Since T is compact by proposition [Xio, proposition 3.55], then the conclusion follows from Fredholm alternative theorem [Xio, theorem 5.1]. □

Remark 9.21. Fredholm alternative [Xio, theorem 5.1] also implies that any eigenspace of a elliptic operator is finite-dimensional. If L is self-adjoint, we can apply spectral decomposition theorem to get more interesting conclusions.

9.I. **Regularity of weak solutions.** In Schauder estimate 9.3 and L^p estimate 9.7, we require u to belong to $C^{2,\alpha}(\overline{\Omega})$ or $W^{2,p}(\Omega)$. These priori estimates ensures the regularity of solutions, as we showed in preceding subsections.

In subsection 9.F, we considered $H^1(\Omega)$-weak solutions. In fact, such weak solutions also have regularity. Specifically, if u is a solution to $Lu = f$ in the sense of $L^1_{\text{loc}}(\Omega)$-weak solution, where f and the coefficients of L have good regularity, then u also have good regularity.

First, we deal with the relatively easy cases, the interior regularity of $H^1(\Omega)$-weak solutions.

Theorem 9.22 (Interior regularity). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. Suppose that L has the divergence form*

$$Lu = -\partial_j \left(a^{ij} \partial_i u \right) + b^i \partial_i u + cu.$$

(1) If

$$a^{ij} \in C^1(\Omega), \quad b^i, c \in L^\infty(\Omega),$$

and if $u \in H^1(\Omega)$ is a weak solution to

$$Lu = f \quad \text{in} \quad \Omega, \quad \text{where} \quad f \in L^2(\Omega)$$

then

$$u \in H^2_{\text{loc}}(\Omega)$$

and for each open subset $U \subset\subset \Omega$, we have the estimate

$$\|u\|_{H^2(U)} \le C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

where $C = C(\Omega, U, a^{ij}, b^i, c)$ is a constant.

(2) If

$$a^{ij}, b^i, c \in C^1 m + 1(\Omega),$$

and if $u \in H^1(\Omega)$ is a weak solution to

$$Lu = f \quad \text{in} \quad \Omega, \quad \text{where} \quad f \in H^m(\Omega)$$

then

$$u \in H^{m+2}_{\text{loc}}(\Omega)$$

and for each open subset $U \subset\subset \Omega$, we have the estimate

$$\|u\|_{H^{m+2}(U)} \le C \left(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

where $C = C(\Omega, U, a^{ij}, b^i, c, m)$ is a constant.

Proof. See [Eva10, section 6.3 — theorems 1, 2]. □

More generally, we introduce the general conclusions.

Theorem 9.23 (Weyl's lemma)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that*

$$a^{ij} \in C^{2,\alpha}(\overline{\Omega}), \quad b_i \in C^{1,\alpha}(\overline{\Omega}), \quad c \in C^1(\overline{\Omega}).$$

Then if u is a $L^1_{\text{loc}}(\Omega)$-weak solution to

$$Lu = f \quad in \quad \Omega, \quad where \quad f \in C^{\alpha}(\overline{\Omega})$$

then u coincides almost everywhere with a function $\widetilde{u} \in C^{2,\alpha}(\overline{\Omega})$ in Ω.

Proof. See [Hel60, section 4.2]. □

Theorem 9.24. *Suppose $\Omega \subset \mathbb{R}^n$ is an open subset, and L is an elliptic operator of order k with smooth coefficients on Ω. Let u and f be distributions on Ω satisfying $Lu = f$. If $f \in H^s_{\text{loc}}(\Omega)$ for some $s \in \mathbb{R}$, then $u \in H^{s+k}_{\text{loc}}(\Omega)$.*

Proof. See [Fol95, theorem 6.33] □

10. Appendix — transfer the results to to compact manifolds

10.A. Differential operators for vector bundles.

Definition 10.1. *Let E, F be smooth vector bundles over a smooth manifold M. We say that a map $L : \Gamma(M, E) \to \Gamma(M, F)$[30] is a **differential operator** if over any affine chart U trivialising both E and F, the map*

$$C^\infty(U, \mathbb{R}^n) \cong \Gamma(U, E) \to \Gamma(U, F) \cong C^\infty(U, \mathbb{R}^m)$$

is a classical differential operator.

*Moreover, L is **linear** if any local expression of L is a classical linear differential operator. The **order** of L is the highest order of all its local expressions.*

*A **differential operator on a manifold** M is a differential operator from $\Gamma(M, M \times \mathbb{R}) \to \Gamma(M, M \times \mathbb{R})$, i.e. a differential operator form $C^\infty(M)$ to $C^\infty(M)$.*

Remark 10.2. (1) The regularity of L is characterized by its local expressions. Unless otherwise stated, the differential operator P is smooth (i.e. all its local expressions are classical smooth differential operators).

(2) If L is linear, then locally, given $(U, (x^s))$ and a local frame (e_i) of E, P is expressed by

$$Lu = L(u^i e_i) = \sum_{|\alpha| \leq m} \partial^\alpha u^i \cdot a^\alpha(e_i) \tag{10.1}$$

where $a^\alpha \in \Gamma(U, \mathrm{Hom}(E, F))$.

(3) By local computation, it's easy to see the linearity is well-defined.

Remark 10.3. In [Kaz16] the author shows that

(1) Generally, for a differential operator, the linearized differential operator may reflect its properties.

(2) For a linear differential operator with variable coefficients, the corresponding linear differential operator with constant coefficients derived by freezing the coefficients at one point, will reflect its properties. (We use the continuity method 9.1.)

An important characterization of linear differential operator is its *principal symbal*, which is motivated by remark 10.3 (2).

Definition 10.4. *Let E, F be smooth vector bundles over a smooth manifold M, and let $L : \Gamma(M, E) \to \Gamma(M, F)$ be a linear differential operator of order m. At any point $p \in M$, and for every $\xi \in T_p^*M$, the **principal symbol** $\sigma_\xi(L; p)$ [or simply $\sigma_\xi(L)$] is defined as follows:*

Given any affine chart $(U, (x^s))$ containing p, then for any local frame (e_i) of E near p, the principal symbol is given by[31]

$$\sigma_\xi(L; p) = \sum_{|\alpha|=m} \xi_\alpha a^\alpha(p) \in \mathrm{Hom}(E_p, F_p)$$

where a^α is given by (10.1) and

$$\xi_\alpha = \xi\left(\frac{\partial}{\partial x^{\alpha_1}}\right) \cdots \xi\left(\frac{\partial}{\partial x^{\alpha_m}}\right).$$

[30] Some people say that L is a morphism between the sheaves of smooth sections of E and F. This is equivalent to that L is a map from $\Gamma(M, E)$ to $\Gamma(M, F)$.

[31] People usually write $\sum_{|\alpha|=m} a^\alpha(p)\xi^\alpha$

Remark 10.5. This definition is well-defined. For any compatible $(V, (y^t))$ and $(\widetilde{e}_i)$,

$$\begin{aligned} Lu &= L(u^i e_i) = \sum_{|\alpha|\le m} \partial_x^\alpha u^i \cdot a^\alpha(e_i) \\ &= L(\widetilde{u}^i \widetilde{e}_i) = \sum_{|\beta|\le m} \partial_y^\beta \widetilde{u}^i \cdot \widetilde{a}^\beta(\widetilde{e}_i). \end{aligned}$$

It follows that

$$\begin{aligned} Lu &= \sum_{|\alpha|=m} \partial_x^\alpha u^i \cdot a^\alpha(e_i) + \text{lower order terms} \\ &= \sum_{|\alpha|=m}\sum_{|\beta|=m} \frac{\partial y^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\beta_m}}{\partial x^{\alpha_m}} \partial_y^\beta u^i \cdot a^\alpha(e_i) + \text{lower order terms} \\ &= \sum_{|\alpha|=m}\sum_{|\beta|=m} \frac{\partial y^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\beta_m}}{\partial x^{\alpha_m}} \partial_y^\beta \widetilde{u}^i \cdot a^\alpha(\widetilde{e}_i) + \text{lower order terms} \\ &= \sum_{|\beta|=m} \left(\partial_y^\beta \widetilde{u}^i \cdot \sum_{|\alpha|=m} \frac{\partial y^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\beta_m}}{\partial x^{\alpha_m}} a^\alpha(\widetilde{e}_i) \right) + \text{lower order terms} \\ &= \sum_{|\beta|=m} \partial_y^\beta \widetilde{u}^i \cdot \widetilde{a}^\beta(\widetilde{e}_i) + \text{lower order terms} \end{aligned}$$

and hence

$$\sum_{|\alpha|=m} \frac{\partial y^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\beta_m}}{\partial x^{\alpha_m}} a^\alpha = \widetilde{a}^\beta.$$

Therefore

$$\begin{aligned} \sum_{|\beta|=m} \xi\left(\frac{\partial}{\partial y^{\beta_1}}\right) \cdots \xi\left(\frac{\partial}{\partial y^{\beta_m}}\right) \widetilde{a}^\beta &= \sum_{|\beta|=m}\sum_{|\alpha|=m} \frac{\partial y^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\beta_m}}{\partial x^{\alpha_m}} \xi\left(\frac{\partial}{\partial y^{\beta_1}}\right) \cdots \xi\left(\frac{\partial}{\partial y^{\beta_m}}\right) a^\alpha \\ &= \sum_{|\alpha|=m} \xi\left(\frac{\partial}{\partial x^{\alpha_1}}\right) \cdots \xi\left(\frac{\partial}{\partial x^{\alpha_m}}\right) a^\alpha \end{aligned}$$

i.e.

$$\sum_{|\beta|=m} \xi_\beta^{(y)} \widetilde{a}^\beta = \sum_{|\alpha|=m} \xi_\alpha^{(x)} a^\alpha,$$

which implies that $\sigma_\xi(L; p)$ is well-defined.

Definition 10.6. *A linear differential operator $P : C^\infty(E) \to C^\infty(F)$ is **elliptic at a point** $x \in M$ if the symbol $\sigma_\xi(P; x)$ is an isomorphism for every $\xi \in T_x^*M \setminus \{0\}$.*

Moreover, for Hermitian vector bundles, we can define the *formal adjoint* of a linear differential operator.

Definition 10.7. *If E and F are smooth Hermitian vector bundles over M and if $P : \Gamma(M, E) \to \Gamma(M, F)$ is a linear differential operator, then one can use the L^2 inner product to define the **formal adjoint**, P^*, by the usual rule*

$$\int \langle Pu, v\rangle_F \, \mathrm{dvol} = \int \langle u, P^* v\rangle_E \, \mathrm{dvol}, \quad \forall u \in C_c^\infty(M, E), \quad \forall v \in C_c^\infty(M, F).$$

Remark 10.8. Since the supports of u and v can be assumed to be in a coordinate patch, one can compute P^* locally using integration by parts.

10.B. **Sobolev spaces and Hölder spaces on manifolds.**

Definition 10.9 (John Lee's version). *Let (M, g) be a Riemannian manifold.*

*(1) Let P be a linear differential operator on M. If u and f are locally integrable functions on M, we say u is a **weak** (or **distribution**) **solution** to the equation $Pu = f$ if*

$$\int_M uP^*\phi \,\mathrm{dvol} = \int_M f\phi \,\mathrm{dvol} \quad \forall \phi \in C_c^\infty(M)$$

where $C_c^\infty(M)$ denotes the set of all compactly supported smooth functions, and P^* is the formal adjoint of P.

*(2) If $q \geq 1$, the **Lebesgue space** $L^q(M)$ is defined by*

$$L^q(M) = \left\{ u \text{ is locally integrable} : \|u\|_q = \left(\int_M |u|^q \,\mathrm{dvol} \right)^{1/q} < \infty \right\}$$

*(3) If $q \geq 1$ and k is a non-negative integer, the **Sobolev space** $L_k^q(M)$ is defined as*[32]

the set of $u \in L^q(M)$ such that $Pu = f \in L^q(M)$ (in the weak sense) whenever P is a smooth differential operator of order $\leq k$.

*We define the **Sobolev norm** $\|\bullet\|_{q,k}$ on $L_k^q(M)$ by:*

$$\|u\|_{q,k} = \left(\sum_{i=0}^{k} \int_M |\nabla^i u|^q \,\mathrm{dvol} \right)^{1/q}$$

where $\nabla^i = \underbrace{\nabla \circ \cdots \circ \nabla}_{i \text{ times}}$.

(4) The space $C^k(M)$ is defined by

$$C^k(M) = \left\{ u \text{ is } k \text{ times continuously differentiable} : \|u\|_{C^k} = \sum_{i=0}^{k} \sup_M |\nabla^i u| < \infty \right\}.$$

*(5) The **Hölder space** $C^{k,\alpha}(M)$ is defined for $0 < \alpha < 1$ by*

$$C^{k,\alpha}(M) = \left\{ u \in C^k(M) : \|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \sup_{x,y} \frac{\left|\nabla^k u(x) - \nabla^k u(y)\right|}{|x-y|^\alpha} < \infty \right\}$$

where the supremum is over all $x \neq y$ such that y is contained in a normal coordinates neighborhood of x, and $\nabla^k u(y)$ is taken to mean the tensor at x obtained by parallel transport along the radial geodesic from x to y.

(6) $C^\infty(M)$ and $C_c^\infty(M)$ denote the spaces of smooth functions and smooth compactly supported functions on M, respectively.

Remark 10.10. The Sobolev space $L_k^q(M)$ is a reflexive Banach space, and $C_c^\infty(M)$ is dense in $L_k^q(M)$.

[32] Some people write $W^{k,p}(M) = L_k^q(M)$.

For compact manifolds, people can simply define Sobolev space and Hölder space via partition of unity.

Definition 10.11 (Kazdan's version). *Let (M, g) be a Riemannian manifold.*

*(1) Let $A \subset \mathbb{R}^n$ be the closure of a connected bounded open set and $0 < \alpha < 1$. Then $f : A \to \mathbb{R}$ is **Hölder continuous with exponent** α if the following expression is finite*

$$[f]_{\alpha,A} = \sup_{\substack{x,y\in A\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

*The **Hölder space** $C^{k,\alpha}(\overline{\Omega})$ is the Banach space of real valued functions f defined on Ω all of whose k-th order partial derivatives are Hölder continuous with exponent α. The norm is*

$$|f|_{k+\alpha} = \|f\|_{C^k(\overline{\Omega})} + \max_{|I|=k}[\partial^I f]_{\alpha,\overline{\Omega}}.$$

On M, one obtains the space $C^{k,\alpha}(M)$ by using a partition of unity. Specifically, let (B_i) be an open cover of M where each (B_i, ϕ_i) is a regular coordinate ball, and let (ρ_i) be a partition of unity subordinate to (B_i); then the norm is

$$\|u\|_{C^{k,\alpha}} = \sum_i \left|(\rho_i u)\circ\phi_i^{-1}\right|_{k+\alpha}$$

*and the **Hölder space** $C^{k,\alpha}(M)$ is*

$$C^{k,\alpha}(M) = \left\{u \text{ is } k \text{ times continuously differentiable} : \|u\|_{C^{k,\alpha}} < \infty\right\}.$$

(2) For $f \in C^\infty(M)$, $1 \le p < \infty$, and an integer $k \ge 0$ define the norm

$$\|f\|_{k,p} = \left(\int_M \sum_{0\le|I|\le k} |\nabla^I f|^p\right)^{1/p}.$$

*The **Sobolev space** $L^p_k(M)$ is the completion of $C^\infty(M)$ in this norm; equivalently, by using local coordinates and partition of unity, one can describe $L^p_k(M)$ as equivalent classes of measurable functions all of whose partial derivatives up to order k are in $L^p(M)$.*

Remark 10.12. For compact manifolds, any open cover has a fintie sub-cover $(U_i)_{i=1}^N$. Suppose each (U_i, ϕ_i) is a coordinate ball. Let $(\rho_i)_{i=1}^N$ be a partition of unity subordinate to (U_i); then

$$\sum_{i=1}^N \left(\int_M \|\rho_i u\|^p \,\mathrm{dvol}\right)^{1/p} \sim \sum_{i=1}^N \left\|(\rho_i u)\circ\phi_i^{-1}\right\|_p,$$

$$N\left(\int_M |u|^p \,\mathrm{dvol}\right)^{1/p} \ge \sum_{i=1}^N \left(\int_M |\rho_i u|^p \,\mathrm{dvol}\right)^{1/p},$$

$$\left(\int_M |u|^p \, \mathrm{dvol}\right)^{1/p} = \left(\int_M \left|\sum_{i=1}^N \rho_i u\right|^p \mathrm{dvol}\right)^{1/p} \leq \left(\sum_{i=1}^N \int_M |\rho_i u|^p \, \mathrm{dvol}\right)^{1/p} \leq \sum_{i=1}^N \left(\int_M |\rho_i u|^p \, \mathrm{dvol}\right)^{1/p}$$

where the red inequality follows from $p \geq 1$. Therefore, for compact manifolds, two definitions coincide (the norms are equivalent).

For non-compact manifolds, two definitions are not equivalent, so people should be careful with the basic definitions. In fact, non-compact situations need special treatment, and we won't discuss them for the time being.[33]

10.C. **Transfer the results to compact manifolds.** In the next we focus on compact manifolds, and we will transfer the results in subsection 8.B and section 9, to a compact manifold M by covering M with small coordinate patches, applying the results in normal coordinates, and summing the results with a partition of unity.

Theorem 10.13 (Sobolev embedding theorem for compact manifolds)**.** *Suppose M is a compact Riemannian manifold of dimension n (possibly with C^1 boundary).*

(1) If

$$\frac{1}{r} \geq \frac{1}{q} - \frac{k}{n},$$

then $L^q_k(M)$ is continuously embedded in $L^r(M)$.

(2) Suppose strict inequality holds in (1). Then the inclusion $L^q_k(M) \subset L^r(M)$ is a compact operator.

(3) Suppose $0 < \alpha < 1$, and

$$\frac{1}{q} \leq \frac{k - \alpha}{n}.$$

Then $L^q_k(M)$ is continuously embedded in $C^\alpha(M)$.

Ahead of giving a proof, we consider the special case, the Sobolev inequality (8.3).

Theorem 10.14 (Aubin)**.** *Let (M, g) be a compact Riemannian manifold, and let σ_n be the best Sobolev constant defined in (8.3). Then for every $\varepsilon > 0$, there exists a constant C_ε with*

$$\|\phi\|_q^2 \leq (1+\varepsilon)\sigma_n \int_M |\nabla \phi|^2 \, \mathrm{dvol} + C_\varepsilon \int_M \phi^2 \, \mathrm{dvol} \quad \forall \phi \in C^\infty(M)$$

where $q = 2n/(n-2)$.

Proof. Fix $\varepsilon > 0$. For each point $p \in M$, we choose a normal coordinates chart $(U, (x^i))$ centered at p such that the eigenvalues of g are between $(1+\varepsilon)^{-1}$ and $(1+\varepsilon)$, and furthermore $dvol = f\, dx$ where $(1+\varepsilon)^{-1} < f < (1+\varepsilon)$. By compactness we choose a finite subcover (U_i) and a subordinate partition of unity, which we may write as (α_i^2),

[33]One can see https://en.wikipedia.org/wiki/Sobolev_inequality for some related discussions.

where $\alpha_i \in C^\infty(M)$ and $\sum_i \alpha_i^2 = 1$. Then we have

$$\begin{aligned} \|\phi\|_q^2 = \|\phi^2\|_{q/2} = \left\| \sum_i \alpha_i^2 \phi^2 \right\|_{q/2} &\leq \sum_i \left(\int_M |\alpha_i \phi|^q \, d\mathrm{vol} \right)^{2/q} \\ &\leq (1+\varepsilon)^{2/q} \sum_i \left(\int_{U_i} |\alpha_i \phi|^q \, dx \right)^{2/q}. \end{aligned}$$

By Sobolev inequality (8.3) and our restriction on the deviation of g and $d\mathrm{vol}$ from the Euclidean metric, we have

$$\left(\int_{U_i} |\alpha_i \phi|^q \, dx \right)^{2/q} \leq \sigma_n \int_{U_i} |d(\alpha_i \phi)|_0^2 \, dx \leq (1+\varepsilon)^2 \sigma_n \int_{U_i} |d(\alpha_i \phi)|^2 \, d\mathrm{vol}$$

where $|\bullet|_0$ denotes the Euclidean metric in normal coordinates. Therefore,

$$\|\phi\|_q^2 \leq (1+\varepsilon)^{2+2/q} \sigma_n \sum_i \int_{U_i} |\nabla(\alpha_i \phi)|^2 \, d\mathrm{vol}.$$

Note that by Hölder inequality and $2ab < \varepsilon a^2 + \varepsilon^{-1} b^2$ we have

$$\begin{aligned} |\nabla(\alpha_i \phi)|^2 &= \alpha_i^2 |\nabla \phi|^2 + 2\alpha_i \phi \langle \nabla \alpha, \nabla \phi \rangle + \phi^2 |\nabla \alpha_i|^2 \\ &\leq (1+\varepsilon)\alpha_i^2 |\nabla \phi|^2 + (1+\varepsilon^{-1}) \phi^2 |\nabla \alpha_i|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|\phi\|_q^2 &\leq (1+\varepsilon)^{2+2/q} \sigma_n \sum_i \int_{U_i} \left((1+\varepsilon)\alpha_i^2 |\nabla \phi|^2 + (1+\varepsilon^{-1}) \phi^2 |\nabla \alpha_i|^2 \right) d\mathrm{vol} \\ &= (1+\varepsilon)^{3+2/q} \sigma_n \int_M |\nabla \phi|^2 \, d\mathrm{vol} + (1+\varepsilon)^{2+2/q} (1+\varepsilon^{-1}) \sigma_n \sum_i \int_{U_i} \phi^2 |\nabla \alpha_i|^2 d\mathrm{vol} \\ &\leq (1+\varepsilon)^{3+2/q} \sigma_n \int_M |\nabla \phi|^2 \, d\mathrm{vol} + C_\varepsilon \int_M \phi^2 d\mathrm{vol} \end{aligned}$$

where the last inequality uses the finiteness and the compactness. By taking ε sufficiently small we get the conclusion. □

Therefore, to a certain degree, the Sobolev inequality holds with the same constant on any compact manifold M.

Moreover, by a technique similar to the proof of theorem 10.14, one can give a proof of the general cases (theorem 10.13) via theorem 8.3.[34] The proof is omitted.

Now we turn to the results for PDE.

Theorem 10.15 (L^p estimate). *Let (M, g) be a compact Riemannian manifold M. Suppose $L = \Delta + c$, where $c \in C^\infty(M)$. Assume $1 < p < \infty$.*

(1) We have the L^p estimate for $L = \Delta + c$:

$$\|u\|_{L^p_{k+2}(M)} \leq C \left(\|Lu\|_{L^p_k(M)} + \|u\|_{L^p(M)} \right) \quad \forall u \in L^p_{k+2}(M). \tag{10.2}$$

[34] Clearly, this technique doesn't work for Poincaré inequality. We will introduce a new method to deal with Poincaré inequality in the next subsection.

(2) Suppose in addition that $c \le 0$ and $c \not\equiv 0$. Then the equation

$$Lu = f \quad \text{[strong solution]} \quad \text{where} \quad f \in L^p(M)$$

has a unique solution $u \in W^{2,p}(M)$. Moreover,

$$\|u\|_{L^p_{k+2}(M)} \le C\,\|Lu\|_{L^p_k(M)} \qquad \forall u \in L^p_{k+2}(M). \tag{10.3}$$

Proof. By a procedure similar to the proof of theorem 10.14, point (1) easily follows from L^p estimate 9.7.

In the next we prove point (2). By lemma 9.17, lemma 9.18 and point (1), we know

$$L^p(M) = L\left(L^p_2(M)\right) \oplus \ker L^*.$$

By the maximum principle [Pet16, theorem 7.1.7], clearly we have $\ker L = 0$ and $\ker L^* = 0$.[35] Therefore, in this case, the bounded linear operator

$$L : W^{2,p}(M) \to L^p(M)$$

is bijective. Then formula (10.3) follows from [Xio, theorem 3.22]. □

Theorem 10.16 (Schauder estimate)**.** *Let (M, g) be a compact Riemannian manifold M. Suppose $L = \Delta + c$, where $c \in C^\infty(M)$.*

(1) We have the Schauder estimate for $L = \Delta + c$,:

$$\|u\|_{C^{k+2,\alpha}(M)} \le C\left(\|Lu\|_{C^{k,\alpha}(M)} + \|u\|_{C^\alpha(M)}\right) \qquad \forall u \in C^{k+2,\alpha}(M). \tag{10.4}$$

(2) Suppose in addition that $c \le 0$ and $c \not\equiv 0$. Then the equation

$$Lu = f \quad \text{where} \quad f \in C^\alpha(M)$$

has a (unique) solution lying in $C^{2,\alpha}(M)$. Moreover,

$$\|u\|_{C^{k+2,\alpha}(M)} \le C\,\|Lu\|_{C^{k,\alpha}(M)} \qquad \forall u \in C^{k+2,\alpha}(M). \tag{10.5}$$

Proof. By a procedure similar to the proof of theorem 10.14, point (1) easily follows from Schauder estimate 9.3.

In the next we prove point (2). Note that $C^\alpha(M) \subset L^p(M)$. By theorem 10.15 (2), for any $f \in C^\alpha(M)$, there exists $u \in W^{2,p}(M)$ such that

$$Lu = f \quad \text{[strong solution]}.$$

By L^p estimate (10.15), we know $u \in L^p_k(M)$ for arbitrarily large k. Then by Sobolev embedding theorem 10.13, $u \in C^\alpha(M)$. Then by Schauder estimate (10.16) we know $u \in C^{2,\alpha}(M)$. Hence we know the bounded linear operator

$$L : C^{2,\alpha}(M) \to C^\alpha(M)$$

is bijective. Then formula (10.5) follows from [Xio, theorem 3.22]. □

Remark 10.17. By L^p estimate (10.2) and Schauder estimate (10.4), for $L = \Delta + c$, if

$$Lu = f \quad \text{where} \quad f \in C^\infty(M)$$

has a solution u in $L^p_2(M)$, then $u \in C^\infty(M)$. Moreover, weak solutions also have regularity. Via subsection 9.I, people can show that:

[35] For general p, one use the definition of adjoint operator and and regularity of weak solutions to guarantee that we can apply the maximum principle for $\ker L^* = 0$.

(1) If $u \in L^1_{\text{loc}}(M)$ is a weak solution to $Lu = f$ and if $f \in L^p_k(M)$, then $u \in L^p_{k+2}(M)$;
(2) If $u \in L^1_{\text{loc}}(M)$ is a weak solution to $Lu = f$ and if $f \in C^{k,\alpha}(M)$, then $u \in C^{k+2,\alpha}(M)$.

More generally, for a compact manifold M, and for a linear elliptic differential operator $P : C^\infty(E) \to C^\infty(F)$ of order k, we also have elliptic estimates (Schauder estimate and L^p estimate), and we also know the existence and regularity of solutions to $Pu = v$. One can refer to [Kaz16, chapter 2].

10.D. **Poincaré inequality on compact manifolds.** To prove the Poincaré inequality, the technique for proving theorem 10.14 doesn't work.

In the next we use the method of Rayleigh quotient to prove the classical version.

Lemma 10.18. *Let (M, g) be a closed Riemannian manifold M. Consider the Laplacian*

$$\Delta : H^2(M) \to L^2(M).$$

Then

$$\operatorname{im}(\Delta) = \left\{ \phi \in L^2(M) : \int_M \phi \, \mathrm{dvol} = 0 \right\}.$$

Proof. By lemma 9.18 and L^p estimate (10.2), we know $\operatorname{im}(\Delta)$ is closed. By lemma 9.17 and the fact that Δ is self-adjoint, we know

$$\operatorname{im}(\Delta) = \ker(\Delta)^\perp.$$

By the maximum principle [Pet16, theorem 7.1.7], $\ker(\Delta) = \operatorname{span}_{\mathbb{R}}\{1\}$. Then

$$\operatorname{im}(\Delta) = 1^\perp = \left\{ \phi \in L^2(M) : \int_M \phi \, \mathrm{dvol} = 0 \right\}.$$

We are done. □

Theorem 10.19. *Let (M, g) be a closed manifold. Consider the eigenvalue problem*

$$-\Delta u = \lambda u. \tag{10.6}$$

Then the eigenvalues of (10.6) can be represented as $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Moreover, given the Rayleigh quotient

$$\mathcal{Q} : H^1(M) \setminus \{0\} \mapsto \mathbb{R}, \quad \phi \mapsto \frac{\int_M |\nabla \phi|^2 \, \mathrm{dvol}}{\int_M \phi^2 \, \mathrm{dvol}}.$$

Then we have

$$\lambda_1 = \mathcal{Q}(u_1) = \inf_{\mathcal{A} \setminus \{0\}} \mathcal{Q} \quad \text{for} \quad \mathcal{A} := \left\{ \phi \in H^1(M) : \int_M \phi \, \mathrm{dvol} = 0 \right\}$$

where u_1 is an eigenfunction corresponding to λ_1.

Remark 10.20. One should pay attention to the choice of $\mathcal{A}$.

(1) We can't choose $\mathcal{A} = H^1(M)$. Otherwise, considering $u \equiv 1$ we know $\inf_{\mathcal{A}\setminus\{0\}} = 0$. Therefore, in this case we get nothing.
(2) Also, we can't choose

$$\mathcal{A} = \{ \phi \in H^1(M) : u \text{ is not constant} \}.$$

It seems like that the variational method shows that the minimizer of $\inf_{\mathcal{A}\setminus\{0\}} \mathcal{Q}$ must be the first eigenvalue and then we get the conclusion. But, in fact, in this case $\inf_{\mathcal{A}\setminus\{0\}} \mathcal{Q}$ doesn't admit a minimizer![36] Hence the approach is wrong.

Proof. Set

$$\mathcal{A}_0 = \left\{u \in L^p(M) : \int_M u\, d\mathrm{vol} = 0\right\} \quad \text{and} \quad \mathcal{A}_1 = \left\{u \in L^p_2(M) : \int_M u\, d\mathrm{vol} = 0\right\}.$$

By lemma 10.18, we know the bounded linear operator

$$\Delta : \mathcal{A}_1 \to \mathcal{A}_0$$

is invertible. Sobolev embedding theorem 10.13 yields that the operator

$$\Delta^{-1} : \mathcal{A}_0 \to \mathcal{A}_1 \hookrightarrow \mathcal{A}_0$$

is compact. Clearly, Δ^{-1} is self-adjoint, $EV(\Delta) \setminus \{0\} = EV(\Delta^{-1}) \setminus \{0\}$, and $0 \notin EV(\Delta^{-1})$.

By Fredholm alternative [Xio, theorem 5.1], each eigenspace is finite-dimensional. Since $L^p(M)$ is infinite-dimensional, [Xio, theorem 5.12] and spectral decomposition [Xio, theorem 5.24] yield that $EV(\Delta^{-1}) = EV(\Delta^{-1}) \setminus \{0\}$ is a sequence converging to 0. Since L^p estimate 10.15 implies that $EV(\Delta) \subset \mathbb{R}_{\geq 0}$, the first assertion easily follows.

In the next we prove the second assertion. First we show that the minimizer of $\mathcal{Q}$ exists. Suppose (ϕ_k) is a sequence in $\mathcal{A} \setminus \{0\}$ such that

$$\mathcal{Q}(\phi_k) \to \inf_{\mathcal{A}\setminus\{0\}} \mathcal{Q}.$$

Setting

$$\psi_k = \frac{\phi_k}{\|\phi_k\|_{L^2(M)}} \quad \forall k,$$

then $\mathcal{Q}(\psi_k) = \mathcal{Q}(\phi_k)$ for each k, and (ψ_k) is a bounded sequence in $H^1(M)$. Since $H^1(M)$ is reflexive, by [Xio, theorem 3.41], there exists a subsequence, which we relabel as (ψ_k), satisfying

$$\psi_k \rightharpoonup u \quad \text{in} \quad H^1(M)$$

for some $u \in H^1(M)$. By Sobolev embedding theorem 10.13, $\psi_k \to u$ in $L^2(M)$, and hence

$$\|u\|_{L^2(M)} = \lim_{k\to\infty} \|\psi_k\| = 1 \quad \text{and} \quad \int_M u\, d\mathrm{vol} = \lim_{k\to\infty} \int_M \psi_k\, d\mathrm{vol} = 0.$$

Moreover,

$$\begin{aligned}\int_M |\nabla u|^2 &= \int_M |\nabla\psi_k|^2 - |\nabla(\psi_k - u)|^2 - 2\langle\nabla(\psi_k - u), \nabla u\rangle \\ &\leq \int_M |\nabla\psi_k|^2 - 2\langle\nabla(\psi_k - u), \nabla u\rangle\end{aligned}$$

[36] Consider $\phi_n = C + \frac{u}{n} \in \mathcal{A} \setminus \{0\}$, where $C \neq 0$ is a constant and $u \in \mathcal{A}$. Then $\mathcal{Q}(\phi_n) \to 0$ but $\phi_n \to C$, where $C \notin \mathcal{A} \setminus \{0\}$.

and hence by $\psi_k \rightharpoonup u$ in $H^1(M)$ we know

$$\int_M |\nabla u|^2 \leq \liminf_k \int_M |\nabla \psi_k|^2.$$

It follows that

$$\mathcal{Q}(u) \leq \liminf_k \mathcal{Q}(\psi_k) = \inf_{\mathcal{A}\setminus\{0\}} \mathcal{Q}$$

Therefore, u is a minimizer of $\inf_{\mathcal{A}\setminus\{0\}} \mathcal{Q}$.

On the other hand, for any $u \in \mathcal{A} \setminus \{0\}$, consider

$$f : \mathbb{R} \times \mathcal{A} \to \mathbb{R}, \quad (t, v) \mapsto \frac{\int_M |\nabla(u+tv)|^2 \,\mathrm{dvol}}{\int_M (u+tv)^2 \,\mathrm{dvol}}.$$

Then

$$f'(t,v) = \frac{\left(2\int_M \langle \nabla u, \nabla v\rangle + 2t\int_M |\nabla v|^2\right)\left(\int_M (u+tv)^2\right) - \left(\int_M |\nabla(u+tv)|^2\right)\left(\int_M 2v(u+tv)\right)}{\left(\int_M (u+tv)^2\right)^2}$$

and hence

$$f'(0,v) = 2\frac{\int_M \langle \nabla u, \nabla v\rangle \cdot \int_M u^2 - \int_M |\nabla u|^2 \cdot \int_M uv}{\left(\int_M u^2\right)^2}.$$

Note that

$$\begin{aligned} f'(0,v) = 0 \quad &\Longleftrightarrow \quad \int_M \langle \nabla u, \nabla v\rangle \cdot \int_M u^2 = \int_M |\nabla u|^2 \cdot \int_M uv \\ &\Longleftrightarrow \quad \int_M (\mathrm{div}\,(v\nabla u) - v\Delta u) = \frac{\int_M |\nabla u|^2}{\int_M u^2} \cdot \int_M uv \\ &\Longleftrightarrow \quad \int_M \left(\frac{\int_M |\nabla u|^2}{\int_M u^2} u + \Delta u\right) v = 0. \end{aligned}$$

Therefore, if u is a minimizer of $\inf_{\mathcal{A}\setminus\{0\}} \mathcal{Q}$, then

$$\int_M \left(\frac{\int_M |\nabla u|^2}{\int_M u^2} u + \Delta u\right) v = 0 \quad \forall v \in \mathcal{A}$$

and hence

$$\frac{\int_M |\nabla u|^2}{\int_M u^2} u + \Delta u \equiv C \quad \text{for some constant } C.$$

Since $u \in \mathcal{A}$, we know

$$\int_M \left(\frac{\int_M |\nabla u|^2}{\int_M u^2} u + \Delta u\right) = 0$$

and hence $C = 0$. Therefore, any minimizer u must be an eigenfunction. Since $u \in \mathcal{A} \setminus \{0\}$, the minimizer u is not constant, and hence u must be an eigenfunction corresponding to λ_k for some $k \geq 1$. Moreover, note that

$$\mathcal{Q}(u) = \frac{\int_M |\nabla u|^2}{\int_M u^2} = \frac{\int_M -u\Delta u}{\int_M u^2} = \lambda_k.$$

Then the second follows from the first assertion. □

Remark 10.21. Furthermore, we can compute λ_k by min-max principle in a similar way.

Remark 10.22. We say that a function $J : W^{1,p}(M) \to \mathbb{R}$ is **weakly lower semi-continuous** on $W^{1,p}(M)$ if

$$J(u) \leq \liminf_{k\to\infty} J(u_k)$$

whenever $u_k \rightharpoonup u$ in $W^{1,p}(M)$. Clearly, by red inequalities, we know that the function

$$J(u) = \int_M |\nabla u|^2 \, \mathrm{dvol}$$

is weakly lower semi-continuous.

Corollary 10.23 (Poincaré inequality)**.** *Let (M, g) be a compact Riemannian manifold of dimension n. Then we have the Poincaré inequality*

$$\|u - u_M\|_{L^2(M)} \leq C\|\nabla u\|_{L^2(M)} \quad \forall u \in H^1(M)$$

where C^{-1} is the first (positive) eigenvalue of (10.6), and

$$u_M = \frac{\int_M u \, \mathrm{dvol}}{\int_M \mathrm{dvol}}.$$

Proof. Just apply theorem 10.19 to the function $u - u_M \in \mathcal{A}$. □

For more general versions of Poincaré inequalities, one can refer to [Li12, section 5] and [Pet16, section 7.1.5].

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