RICCI FLOW AND THE SPHERE THEOREM

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ABSTRACT. In this paper, we give a succinct and comprehensible introduction to the basic theory of Ricci flow, including the short time existence and uniqueness, Hamilton's convergence criterion, and the differentiable sphere theorem.

CONTENTS

1. Introduction

In 1904, Poincaré proposed his famous conjecture: every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. From the point of view of results, Hamilton's work in 1982 solved some special cases of Poincaré conjecture, and Perelman's work in 2002 developed Hamilton's method and completed the proof.

The basic idea of Hamilton's work [5] is to meliorate some initial metric on the manifold by an evolution equation, which is called the Ricci flow. Hamilton showed that if the initial metric enjoys positive Ricci curvature, then after a rescaling, the Ricci flow will converge to a metric with constant curvature. It is well-known that a simply connected Riemannian manifold with positive constant curvature is diffeomorphic to the sphere, and hence Hamilton provided a effective procedure for proving Poincaré conjecture. However, Hamilton's hypothesis is too strong for Poincaré conjecture, since for general initial metrics, the Ricci flow must lead to more complicated singularities.

Perelman [7] [8] [9] then made a significant contribution to understanding the singularities. He showed that if one takes a certain perspective, the singularities appearing in fintie time can only look like shrinking spheres or cylinders. Moreover, Perelman showed that we can cut the manifold along the singularities, devide the manifold into several pieces, and then continue the Ricci flow on each piece. For a closed 3-manifold, Perelman indicated that the above process deforms the manifold into round pieces with strands running between them; moreover, we can rebuild the original manifold by connecting the spheres together with three-dimensional cylinders and see that the original manifold is homeomorphic to the sphere.

However, the original work of Hamilton and Perelman is devoted to solving very general situations and uses quite complicated techniques. For people who know the basic knowledge of Riemannian geometry but are not familiar with these specific techniques, it is difficult for them to get a clear picture of their brilliant work. In fact, some complicated techniques can be simplified, and if one only focuses on the main results, the theory can be organized in a more comprehensible way.

For instance, DeTurck [4] introduced a novel proof of the local existence of the Ricci flow. Instead of using the Nash-Moser implicit function theorem as in Hamilton [5], which is powerful but elaborate, DeTurck proposed an elementary proof that only utilizes the basic transformations. His idea is to show that under some fundamental transformations, the evolution equation is equivalent to some strictly parabolic equation system, which is a classical and well-known object. However, even the simplified proof of DeTurck [4] can be further simplified, and we will show this later.

In this paper, we aim to provide a succinct and comprehensible account of the existence and coverngence theory for the Ricci flow. This theory is an essential part of Hamilton and Perelman's work, which can be regarded as the first step to understand their work. Moreover, this theory itself is self-contained and substantial. We will introduce some major consequences of this theory, such as the differentiable sphere theorem: if a closed Riemannian manifold is 1/4-pinched, then it is diffeomorphic to a spherical space form.

To reorganize this theory in a succinct and comprehensible way, we will extract the core skeleton of this theory and focus on the motivations and ideas of every step in the process of establishing this theory. The reader is only assumed to be familiar with basic Riemannian geometry, and many claims of Riemannian geometry will be left to readers. This is like saying that the theory is a beef cattle, the reader is a cook who can handle the meat on the chopping board, and the author just does the job of dividing the beef cattle into pieces that every cook can handle.

2. THE ORIGIN OF RICCI FLOW AND THE FRAMEWORK OF THIS PAPER

In Hamilton [5], he introduced Ricci flow to prove the following result.

Theorem 2.1. *Let X be a closed 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then X also admits a metric of constant positive curvature.*

We roughly discourse Hamilton's reason for proposing Ricci flow as follows.

To find the desired metric, the basic idea is to derive a flow of metrics $g(t)$ that converges to the desired metric $\overline{g}$ via some parabolic equation

$$\frac{\partial}{\partial t}g_{ij} = P(g)_{ij} \quad \text{where } P \text{ is some elliptic linear differential operator.}$$

Assume our idea can be achieved if we choose P appropriately. Then one can hope that $P(\overline{g}) = 0$. Considering our goal (theorem 2.1), P should relate to the Ricci curvature.

Since $\mathrm{Ric}(g)$ is **nearly** elliptic with respect to g, we may try the simplest case $P = c\cdot\mathrm{Ric}$; although this model doesn't satisfy $P(\overline{g}) = 0$, but its normalized edition will, and two editions are equivalent.

Definition 2.2. *Let (M^n, g) be a closed Riemannian manifold.* ***Hamilton's Ricci flow*** *is the evolution equation*

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}, \tag{2.1}$$

and ***the normalized Hamilton's Ricci flow*** *is the evolution equation*

$$\frac{\partial}{\partial t}g_{ij} = 2\left(\frac{r}{n}g_{ij} - R_{ij}\right) \quad \text{where} \quad r = \frac{\int_M R\,\mathrm{dvol}}{\int_M \mathrm{dvol}}. \tag{2.2}$$

Remark 2.3. The factor r serves to normalize the equation so that the volume is constant. To see this, note that by Jacobi's formula we know

$$\frac{\partial}{\partial t}\log\sqrt{\det(g_{ij})} = \frac{1}{2}g^{ij}\frac{\partial}{\partial t}g_{ij} = r - R,$$

and hence

$$\frac{\partial}{\partial t}\int_M \mathrm{dvol}_g = \int_M \left(\frac{\partial}{\partial t}\log\sqrt{\det(g_{ij})}\right)\mathrm{dvol}_g = \int_M (r-R)\,\mathrm{dvol}_g = 0.$$

Proposition 2.4. *These two evolution equations (2.1) and (2.2) are equivalent.*

Proof. Let t, g_{ij}, R_{ij}, R, r denote the variables for the unnormalized equation, and $\widetilde{t}$, $\widetilde{g}_{ij}$, $\widetilde{R}_{ij}$, $\widetilde{R}$, $\widetilde{r}$ the corresponding variables for the normalized equation.

To make the conversion from (2.1) to (2.2), we choose the normalization factor $\psi(t)$ so that if $\widetilde{g}_{ij} = \psi g_{ij}$ then $\int \mathrm{dvol}_{\widetilde{g}} = 1$, and choose a new time scale $\widetilde{t} = \int \psi(t)\,dt$. Clearly,

$$\int_M \mathrm{dvol}_g = \psi^{-n/2} \quad \text{and} \quad \frac{\partial}{\partial \widetilde{t}} = \frac{1}{\psi(t)}\frac{\partial}{\partial t}.$$

Applying the evolution equation we know

$$\frac{\partial}{\partial t}\log\sqrt{\det(g_{ij})} = \frac{1}{2}g^{ij}\frac{\partial}{\partial t}g_{ij} = -R \quad \text{and hence} \quad \frac{\partial}{\partial t}\log\int_M \mathrm{dvol}_g = -r.$$

Therefore, $\frac{d}{dt}\log\psi = \frac{2}{n}r$, and it easily follows that

$$\frac{\partial}{\partial\tilde{t}}\tilde{g}_{ij} = \frac{\partial}{\partial t}g_{ij} + \left(\frac{d}{dt}\log\psi\right)g_{ij} = \frac{2}{n}\tilde{r}\tilde{g}_{ij} - 2\tilde{R}_{ij}.$$

Clearly, this conversion gives the equivalence. □

Remark 2.5. The normalized Ricci flow is derived by rescaling the unnormalized Ricci flow and rescaling the time.

There are two core steps in Hamilton's framework:

(1) First, we prove the short time existence and uniqueness for the initial-value problem about (2.1).

(2) Second, applying the maximum principle, we get apriori estimates and then some agreeable convergence of the normalized Ricci flow follows.

Remark 2.6. Moreover, if we know the maximum time for Ricci flow, instead of showing the convergence of the normalized Ricci flow, we can also directly show that after rescaling the unnormalized Ricci flow we will get a convergent flow of metrics.

In this project, we will simplify the work of Hamilton [5]. The framework of this project is as follows:

(1) First, we will give a succinct proof of the short time existence and uniqueness, which simplifies the work of Hamilton [5] and DeTurck [4]. (Section 3.)

(2) Next, we will re-establish Hamilton's convergence criterion for the Ricci flow in a more comprehensible way as we said in remark 2.6. (Sections 4, 5, 6.)

(3) Finally, as an important application, we will prove the differentiable sphere theorem. (Section 7.)

3. Short time existence and uniqueness of Hamilton's Ricci flow

3.A. **Short time existence and uniqueness of parabolic equations.** To show the short time existence and uniqueness, we will apply the parabolic theory.

Definition 3.1. *Let F be a vector bundle over a manifold M, and let $L : C^\infty(M,F) \to C^\infty(M,F)$ be a differential operator. Then the* ***linearization*** *of L around any $f \in C^\infty(M,F)$ is defined by*

$$DL_f(g) = \frac{d}{dt}\Big|_{t=0} L(f+tg).$$

Theorem 3.2. *Let F be a vector bundle over a manifold M, let $L : C^\infty(M,F) \to C^\infty(M,F)$ be a differential operator of order 2, and let A be an open set in F. If the restriction (of an open set in a Fréchet space to itself)*

$$L : C^\infty(M,A) \subset C^\infty(M,F) \to C^\infty(M,F)$$

is parabolic, i.e. the linearization of L is parabolic around any $f \in C^\infty(M,A)$, then the evolution equation

$$\frac{\partial f}{\partial t} = L(f)$$

has a unique smooth solution for any initial value problem $f(0) = f_0 \in C^\infty(M,A)$ for at least a short time interval $0 \le t \le \varepsilon$ (where ε may depend on f_0).

Proof. Standard. □

3.B. **Weak parabolicity.** Hamilton's Ricci flow (2.1) is weakly parabolic.

Let $\Sigma^2_+ T^*M$ be the bundle of positive definite symmetric $(0,2)$-tensors, which is an open subset of the bundle $\Sigma^2 T^*M$. We need to show that the linearization of $-2\mathrm{Ric} : \Gamma(\Sigma^2_+ T^*M) \to \Gamma(\Sigma^2 T^*M)$ around any $g \in \Gamma(\Sigma^2_+ T^*M)$ is weakly elliptic.

Proposition 3.3. *Assume that $\frac{\partial}{\partial t} g_{ij} = h_{ij}$, where $g(t)$ is a smooth family of Riemannian metrics and $h(t)$ is a smooth family of symmetric $(0,2)$-tensors. Then*

$$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} h_{kl} \tag{3.1}$$

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \right) \tag{3.2}$$

$$\frac{\partial}{\partial t} R_{ijk}{}^l = \frac{1}{2} g^{lp} \left(\nabla_i \nabla_k h_{jp} + \nabla_j \nabla_p h_{ik} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_k h_{ip} - R_{ijk}{}^q h_{qp} - R_{ijp}{}^q h_{kq} \right) \tag{3.3}$$

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{pq} \left(\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp} \right) \tag{3.4}$$

$$\frac{\partial}{\partial t} R = -\Delta(\mathrm{tr}_g h) + \nabla^p \nabla^q h_{pq} - \langle h, \mathrm{Ric} \rangle \tag{3.5}$$

Corollary 3.4. *The linearization $D(-2\mathrm{Ric})_g : \Gamma(\Sigma^2 T^*M) \to \Gamma(\Sigma^2 T^*M)$ satisfies*

$$D(-2R_{ij})_g(h) = g^{pq}\partial_p\partial_q h_{ij} + g^{pq}\partial_i\partial_j h_{pq} - g^{pq}\partial_p\partial_j h_{iq} - g^{pq}\partial_i\partial_q h_{pj} + \mathrm{LOT}^{\le 1}(h), \tag{3.6}$$

and hence given $\xi \in T_p^*M$ we have

$$\sigma_\xi\left(D(-2\mathrm{Ric})_g\right)(T_{ij}) = g^{pq}\xi_p\xi_q T_{ij} + g^{pq}\xi_i\xi_j T_{pq} - g^{pq}\xi_p\xi_j T_{iq} - g^{pq}\xi_i\xi_q T_{pj}.$$

Since the principal symbol is independent from the choice of local coordinates, WLOG we assume that $g_{ij}(p) = \delta_{ij}$, $\xi_1 = 1$ and $\xi_i = 0$ for any $i \neq 1$. Then

$$\left[\sigma_\xi\left(D(-2\mathrm{Ric})_g\right)(T)\right]_{ij} = \begin{cases} \sum_{k=2}^n T_{kk} & i = j = 1, \\ 0 & i = 1,\ j \neq 1, \\ 0 & i \neq 1,\ j = 1, \\ T_{ij} & i \neq 1,\ j \neq 1. \end{cases}$$

Thus one can see that $D(-2\mathrm{Ric})_g$ is weakly (but not strictly) elliptic, and hence (2.1) is weakly (but not strictly) parabolic.

One can also indicate that (2.1) is not strictly parabolic by the fact that the solutions of the steady state equation $\mathrm{Ric}(g) = 0$ are invariant under the full diffeomorphism group, which is infinite dimensional.

Remark 3.5. The diffeomorphism invariance of the Riemannian curvature tensor will imply the Bianchi identites. For example, linearizing the equation $R\left(\phi_t^* g\right) = \phi_t^*\left(R(g)\right)$ we get $DR_g(L_X g) = XR$; then by (3.5) we derive the contracted second Bianchi identity.

3.C. **DeTurck's trick.** To show the short time existence and uniqueness of Hamilton's Ricci flow, DeTurck introduced a clever way: we turn to solve the evolution equation for $\overline{g}_t = \phi_t^* g_t$, where $\{\phi_t\}$ is a family of diffeomorphisms.

This trick about time, which is related to the reason of weak parabolicity, may influence parabolicity of the evolution equation.

Lemma 3.6. *Let $\{\phi_t\}$ be a smooth family of diffeomorphisms, and let $\{\theta_t\}$ be a smooth family of $(0,2)$-tensors. If $\eta_t = \phi_t^*\theta_t$, then*

$$\frac{\partial}{\partial t}\eta_t = \phi_t^*\left(\mathcal{L}_{\frac{\partial \phi_t}{\partial t}}\theta_t + \frac{\partial \theta_t}{\partial t}\right) = \mathcal{L}_{X_t}\eta_t + \phi_t^*\frac{\partial \theta_t}{\partial t} \quad \text{where} \quad X_t = \left(\phi_t^{-1}\right)_* \frac{\partial \phi_t}{\partial t}.$$

Therefore, if g_t solves Hamilton's Ricci flow, then $\overline{g}_t := \phi_t^* g_t$ satisfies

$$\frac{\partial}{\partial t}\overline{g}_t = \mathcal{L}_{X_t}\overline{g}_t + \phi_t^*\frac{\partial g_t}{\partial t} = \mathcal{L}_{X_t}\overline{g}_t - 2\mathrm{Ric}(\overline{g}_t) \quad \text{where} \quad X_t = \left(\phi_t^{-1}\right)_* \frac{\partial \phi_t}{\partial t}.$$

We can choose $\{\phi_t\}$ appropriately such that $X_t = Y(\overline{g})$.

Lemma 3.7. *Let $\omega : \Gamma(\Sigma^2 T^*M) \to \Gamma(T^*M)$ be a linear differential operator, and let $\tau : \Gamma(\Sigma_+^2 T^*M) \to \Gamma(T^*M)$ be a differential operator. Given any $g \in \Gamma(\Sigma_+^2 T^*M)$, if $\omega \sim (D\tau)_g$,*[1] *then $\nabla_i\omega_j + \nabla_j\omega_i \sim D\left(\mathcal{L}_{\tau(g)^\flat} g\right)_g$.*

Proposition 3.8. *If we set*

$$Y : \Gamma(\Sigma_+^2 T^*M) \to \Gamma(TM), \quad g_{ij} \mapsto g^{pq}(\Gamma_{pq}^r - \widehat{\Gamma}_{pq}^r)$$

where $\widehat{\Gamma}_{pq}^r$ is the Christoffel symbol of some fixed metric $\widehat{g}$, then

$$D\left(\mathcal{L}_{Y(g)} g - 2\mathrm{Ric}\right)_g(h) = \Delta h_{ij} + \mathrm{LOT}^{\leq 1}(h).$$

[1]For two linear differential operators $L_1, L_2 : C^\infty(M,E) \to C^\infty(M,F)$, we say $L_1 \sim L_2$ if $\deg(L_1) = \deg(L_2)$ and $\deg(L_1 - L_2) < \deg(L_1)$.

Proof. Note that we can rewrite $D(-2\mathrm{Ric})_g$ as

$$D(-2R_{ij})_g(h) = \Delta h_{ij} - \nabla_i \omega_j - \nabla_j \omega_i + \mathrm{LOT}^{\leq 0}(h)$$

where

$$\omega_k = \frac{1}{2} g^{pq} \left(\nabla_p h_{qk} + \nabla_q h_{pk} - \nabla_k h_{pq} \right) = D(g_{kr} Y^r)_g(h).$$

Then the conclusion follows from lemma 3.7. □

Corollary 3.9. *If g_t solves Hamilton's Ricci flow in a short time, if $\widehat{\Gamma}^r_{pq}$ is the Christoffel symbol of some fixed metric $\widehat{g}$, and if ϕ_t satisfies*

$$\left[\left(\phi_t^{-1}\right)_* \frac{\partial \phi_t}{\partial t} \right]^m = \overline{g}^{pq} \left(\overline{\Gamma}^m_{pq} - \widehat{\Gamma}^m_{pq} \right) \quad \textit{where} \quad \overline{g}_t = \phi_t^* g_t \tag{3.7}$$

then the evolution equation of $\overline{g}_t$ is

$$\frac{\partial}{\partial t} \overline{g}_{ij} = \overline{\nabla}_i Y_j + \overline{\nabla}_j Y_i - 2\overline{R}_{ij} \quad \textit{where} \quad Y^m = \overline{g}^{pq} \left(\overline{\Gamma}^m_{pq} - \widehat{\Gamma}^m_{pq} \right), \tag{3.8}$$

which is strictly parabolic. We call (3.8) the ***DeTurck's Ricci flow****.*

Remark 3.10. Clearly by the proof of proposition 3.8 we know that DeTurck's Ricci flow is strictly parabolic.

Remark 3.11. Setting $\psi_t = \phi_t^{-1}$, then ψ_t will satisfy a harmonic map flow, which is parabolic. Moreover, one can apply this fact to showing the uniqueness of Hamilton's Ricci flow. We will introduce this in subsection 3.D.

The process from DeTurck's Ricci flow to Hamilton's Ricci flow is much easier. One should note that this process implies the short time existence of Hamilton's Ricci flow.

Theorem 3.12. *If $\overline{g}_t$ solves DeTurck's Ricci flow in a short time, then one can define a smooth family of diffeomorphisms ψ_t by*

$$\frac{\partial \psi_t}{\partial t} = -Y_t \quad \textit{and} \quad \psi_0 = \mathrm{id}, \quad \textit{where} \quad Y^m = \overline{g}^{pq} \left(\overline{\Gamma}^m_{pq} - \widehat{\Gamma}^m_{pq} \right). \tag{3.9}$$

Moreover, $g_t := \psi_t^* \overline{g}_t$ solves Hamilton's Ricci flow in a short time.

Proof. The existence of ψ_t follows from the standard ODE theory. Then lemma 3.6 yields

$$\frac{\partial}{\partial t} g_t = \psi_t^* \left(\mathcal{L}_{-Y_t} \overline{g}_t + \frac{\partial \overline{g}_t}{\partial t} \right) = \psi_t^* \left(-2\mathrm{Ric}\left(\overline{g}_t \right) \right) = -2\mathrm{Ric}\left(g_t \right).$$

Hence g_t solves Hamilton's Ricci flow in a short time. □

Corollary 3.13. *Given any initial data, we can slove Hamilton's Ricci flow in a short time.*

Proof. It follows from theorems 3.2 and 3.12. □

3.D. **The harmonic map flow.** Suppose that g_t solves Hamilton's Ricci flow in a short time. Now we come back to equation (3.7) and show the existence of such $\{\phi_t\}$.

Lemma 3.14. *If $\{\phi_t\}$ is a smooth family of diffeomorphisms, and if $\psi_t = \phi_t^{-1}$, then*

$$\left(\phi_t^{-1}\right)_* \frac{\partial \phi_t}{\partial t} = -\frac{\partial \psi_t}{\partial t}.$$

Clearly, equation (3.7) is equivalent to

$$\frac{\partial \psi_t}{\partial t} = -\overline{g}^{pq}\left(\overline{\Gamma}^m_{pq} - \widehat{\Gamma}^m_{pq}\right) \quad \text{where} \quad \overline{g}_t = \left(\psi_t^{-1}\right)^* g_t \tag{3.10}$$

In the next we will show that equation (3.10) is a harmonic map flow, which is parabolic, and hence the existence follows.

Lemma 3.15. *If $f : (M, g) \to (N, h)$ is a smooth map, then*[2]

$$\left(\Delta_{g,h} f\right)^\gamma = g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}(\Gamma_h)^\alpha_{\beta\gamma}\circ f - (\Gamma_g)^k_{ij}\frac{\partial f^\alpha}{\partial x^k}\right) \tag{3.11}$$

If $f : (M, g) \to (N, h)$ is, in addition, a diffeomorphism, then

$$\left(\Delta_{g,h} f\right)^\gamma = -\tilde{g}^{\alpha\beta}\left[(\Gamma_{\tilde{g}})^\gamma_{\alpha\beta}\circ f - (\Gamma_h)^\gamma_{\alpha\beta}\circ f\right] \quad \textit{where} \quad \tilde{g} = (f^{-1})^* g. \tag{3.12}$$

Corollary 3.16. *Equation (3.10) is equivalent to*

$$\frac{\partial \psi_t}{\partial t} = \Delta_{g_t, \widehat{g}}\, \psi_t. \tag{3.13}$$

Moreover, equation (3.13) is parabolic.

Remark 3.17. We call (3.13) together with $\psi_0 = \mathrm{id}$ a **harmonic map flow**.

Theorem 3.18. *A solution of the Ricci flow is uniquely determined by its initial data.*

Proof. Fix a background metric $\widehat{g}$. Suppose that $g_1(t)$ and $g_2(t)$ solve Hamilton's Ricci flow with $g_1(0) = g_2(0)$.

By corollary 3.16 and theorem 3.2, for $i = 1, 2$, let $\psi_i(t)$ be the solution of harmonic map flow with respect to $g_i(t)$ and $\widehat{g}$. Then both $\overline{g}_i(t) = \left(\psi_i^{-1}\right)^* g_i(t)$ solve DeTurck's Ricci flow with $\overline{g}_1(0) = \overline{g}_2(0)$.

Since DeTurck's Ricci flow is parabolic, by theorem 3.2 we know $\overline{g}_1(t) = \overline{g}_2(t)$ for as long as both exist, and then both $\psi_i(t)$ are solutions of the ODE

$$\frac{\partial \psi_i(t)}{\partial t} = -Y_t \quad \text{where} \quad Y^m = \overline{g}^{pq}\left(\overline{\Gamma}^m_{pq} - \widehat{\Gamma}^m_{pq}\right).$$

Therefore, $\psi_1(t) = \psi_2(t)$ for as long as they are both defined, which implies

$$g_1(t) = \psi_t^* \overline{g}_1(t) = \psi_t^* \overline{g}_2(t) = g_2(t).$$

We are done. □

[2] Recall some basic concepts: for a smooth map $f : (M, g) \to (N, h)$, the second fundamental form $B \in \Gamma(M, T^*M \otimes T^*M \otimes f^*TN)$ is given by

$$B(X, Y) := \widehat{\nabla}_X (f_* Y) - f_*\left(\nabla^M_X Y\right) \in \Gamma(M, f^*TN),$$

and the Laplacian is given by $\Delta_{g,h} f = \mathrm{tr}_g B \in \Gamma(M, f^*TN)$.

4. The maximal time and derivative estimates

This section is a preliminary of Hamilton's maximum principle and Hamilton's convergence theory.

4.A. The finite-time explosion.

First we show that the scalar maximum principle Referencesthm scalar maximum principle implies that the finite-time explosion of Ricci flow with initial strictly positive scalar curvature.

Theorem 4.1. *Suppose $g(t)$ solves Ricci flow on a closed manifold M^n, defined for $t \in [0, T)$. If the metric $g(0)$ has strictly positive scalar curvature, then $g(t)$ becomes singular in finite time, i.e. $T < \infty$.*

Proof. By formula (3.5) we know

$$\frac{\partial}{\partial t}R = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{n}R^2.$$

Assume $R(0)$ is bounded below by some $\rho > 0$. The solution to the associated ODE

$$\frac{d\phi}{dt} = \frac{2}{n}\phi^2$$

with $\phi(0) = \rho$ is

$$\phi(t) = \frac{\rho n}{n - 2t\rho}.$$

Then theorem 8.1 yields that $R(x, t)$ becomes singular in finite time. So the metric becomes singular in finite time. □

4.B. Evolution equations for derivatives of curvature.

Definition 4.2 ($*$-notation)**.** *Given two tensors A, B on a Riemannian manifold M^n, we denote by $A * B$ any quantity obtained from $A \otimes B$ by one or more of these operations:*

(1) summation over pairs of matching upper and lower indices,
(2) contraction on upper indices with respect to the metric,
(3) contraction on lower indices with respect to the metric inverse,
(4) multiplication by constants depending only on n and the ranks of A and B.

Lemma 4.3. *Suppose that $g(t)$ solves Hamilton's Ricci flow on M. Let $A(t)$ and $F(t)$ be two smooth families of tensor fields of the same type. If it holds that*

$$\frac{\partial}{\partial t}A = \Delta_{g(t)}A + F \tag{4.1}$$

then

$$\frac{\partial}{\partial t}\nabla A = \Delta(\nabla A) + \nabla F + \mathrm{Rm} * \nabla A + \nabla \mathrm{Ric} * A$$

and

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + F * A + \mathrm{Ric} * A^{*2}.$$

Theorem 4.4. *The evolution equation for the k-th iterated covariant derivative of the Riemannian curvature tensor under the Ricci flow is*

$$\frac{\partial}{\partial t}\nabla^k \mathrm{Rm} = \Delta\nabla^k \mathrm{Rm} + \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm} \tag{4.2}$$

and the square of the norm of k-th iterated covariant derivative of the Riemannian curvature tensor satisfies the heat-type equation

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{Rm}|^2 = \Delta|\nabla^k \mathrm{Rm}|^2 - 2|\nabla^{k+1}\mathrm{Rm}|^2 + \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm} * \nabla^k \mathrm{Rm}. \tag{4.3}$$

Proof. One can compute that

$$\frac{\partial}{\partial t}\mathrm{Rm} = \Delta \mathrm{Rm} + \mathrm{Rm}^{*2}. \tag{4.4}$$

Thus (4.2) holds for $k = 0$. If (4.2) holds for $k = m$, then applying lemma 4.3 to

$$A = \nabla^m \mathrm{Rm} \quad \text{and} \quad F = \sum_{j=0}^{m} \nabla^j \mathrm{Rm} * \nabla^{m-j}\mathrm{Rm}$$

we know

$$\begin{aligned}\frac{\partial}{\partial t}\nabla\nabla^m \mathrm{Rm} &= \Delta\nabla\nabla^m \mathrm{Rm} + \nabla F + \mathrm{Rm} * \nabla\nabla^m \mathrm{Rm} + \nabla \mathrm{Ric} * \nabla^m \mathrm{Rm} \\ &= \Delta\nabla^{m+1}\mathrm{Rm} + \sum_{j=0}^{m+1} \nabla^j \mathrm{Rm} * \nabla^{m+1-j}\mathrm{Rm}.\end{aligned}$$

Hence we get (4.2) by induction. Now for each k we can truly apply lemma 4.3 to

$$A = \nabla^k \mathrm{Rm} \quad \text{and} \quad F = \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm},$$

which derives that

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k \mathrm{Rm}|^2 &= \Delta|\nabla^k \mathrm{Rm}|^2 - 2|\nabla\nabla^k \mathrm{Rm}|^2 + F * \nabla^k \mathrm{Rm} + \mathrm{Ric} * (\nabla^k \mathrm{Rm})^{*2} \\ &= \Delta|\nabla^k \mathrm{Rm}|^2 - 2|\nabla^{k+1}\mathrm{Rm}|^2 + \sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm} * \nabla^k \mathrm{Rm}.\end{aligned}$$

We are done. □

4.C. **Derivative estimates for Riemannian curvature tensor.**

Theorem 4.5 (Bernstein-Bando-Shi). *If $g(t)$ solves Ricci flow on a closed manifold M^n, then for each $\alpha > 0$ and $m \in \mathbb{N}$, there exists a constant $C_m = C_m(m, n, \max\{\alpha, 1\})$ such that if*

$$|\mathrm{Rm}_{g(t)}| \le \beta \quad \forall t \in \left[0, \frac{\alpha}{\beta}\right],$$

then

$$|\nabla^m \mathrm{Rm}_{g(t)}| \le \frac{C_m \beta}{t^{m/2}} \quad \forall t \in \left(0, \frac{\alpha}{\beta}\right].$$

Proof. We prove by induction. For $m = 0$ the result is just the hypothesis. Assume the result is true for all $m \le k-1$. Note that

$$\begin{aligned}\sum_{j=0}^{k} \nabla^j \mathrm{Rm} * \nabla^{k-j}\mathrm{Rm} * \nabla^k \mathrm{Rm} &\le \sum_{j=0}^{k} c_{kj}|\nabla^j \mathrm{Rm}||\nabla^{k-j}\mathrm{Rm}||\nabla^k\mathrm{Rm}| \\ &\le C'_k\beta|\nabla^k\mathrm{Rm}|^2 + \frac{C''_k}{t^{k/2}}\beta^2|\nabla^k\mathrm{Rm}| \\ &\le \overline{C}_k\beta|\nabla^k\mathrm{Rm}|^2 + \frac{\beta^3}{t^k}.\end{aligned}$$

Therefore by formula (4.3) we know[3]

$$\frac{\partial}{\partial t}|\nabla^k\mathrm{Rm}|^2 \le \Delta|\nabla^k\mathrm{Rm}|^2 + \overline{C}_k\beta|\nabla^k\mathrm{Rm}|^2 + \frac{\beta^3}{t^k}. \tag{4.5}$$

Namely, setting $u_m(t) = t^m|\nabla^m\mathrm{Rm}|^2$ for each m, we have

$$\frac{\partial u_k}{\partial t} \le \Delta u_k + \left(\overline{C}_k\beta + \frac{k}{t}\right)u_k + \beta^3. \tag{4.6}$$

The associated ODE of (4.6) can not be solved near 0 since we have the bad term $\frac{k}{t}u_k$, and hence we can not derive an estimate of u_k directly. We turn to evaluate the new quantity[4]

$$u = u_k + \sum_{m=0}^{k-1}\xi_{km}u_m.$$

In hope that the summation will bring a nice associated ODE, for $m < k$ we compute

$$\begin{aligned}\frac{\partial u_m}{\partial t} &= \Delta u_m + \frac{m}{t}u_m - \frac{2}{t}u_{m+1} + t^m\sum_{j=0}^{m}\nabla^j\mathrm{Rm} * \nabla^{m-j}\mathrm{Rm} * \nabla^m\mathrm{Rm} \\ &\le \Delta u_m + \frac{m}{t}u_m - \frac{2}{t}u_{m+1} + \overline{C}_m\beta^3.\end{aligned}$$

Clearly we can choose $\xi_{km} = \xi_{km}\left(m, \xi_{k,k-1}\right)$ $(0 \le m \le k-2)$ appropriately such that

$$\frac{\partial u}{\partial t} \le \Delta u + \left(\overline{C}_k\beta + \frac{k}{t} - \frac{2}{t}\xi_{k,k-1}\right)u_k + \widetilde{C}(\xi_{k,k-1}, k, n)\beta^3.$$

Now choose

$$\xi_{k,k-1} \ge \frac{\overline{C}_k\alpha + k}{2} \implies \overline{C}_k\beta + \frac{k}{t} - \frac{2}{t}\xi_{k,k-1} \le 0 \quad \forall t \in \left(0, \frac{\alpha}{\beta}\right].$$

Then clearly[5]

$$\frac{\partial u}{\partial t} \le \Delta u + \widetilde{C}_k\beta^3 \quad \forall t \in \left[0, \frac{\alpha}{\beta}\right], \quad \text{where} \quad \widetilde{C}_k = \widetilde{C}_k(k, n, \max\{\alpha, 1\}).$$

[3]We discard the non-positive term $-2|\nabla^{k+1}Rm|^2$ in (4.3), since this term of highest degree does not fit theorem 8.1 whenever we apply theorem 8.1 to $|\nabla^k\mathrm{Rm}|^2$ or any natural quantity related to it.

[4]In order to evaluate u_k, it's equivalent to estimating u since we alredy know a bound on each u_j $(j < k)$. However, the new quantity may satisfy a new parabolic inequality which enjoys a nice associated ODE.

[5]If one replaces u_m/t by $t^{m-1}|\nabla^m R|^2$ for $m \ge 1$, and identifies $\left(\frac{m}{t}u_m\right)\big|_{m=0}$ with 0, then one knows clearly the inequality at $t = 0$.

Since $u(0) = \xi_{k0}|\mathrm{Rm}|^2_{g(0)} \le \xi_{k0}\beta^2$, theorem 8.1 yields that

$$\sup_{x\in M} u(x,t) \le \xi_{k0}\beta^2 + \widetilde{C}_k\beta^3 t.$$

Since ξ_{k0} and $\widetilde{C}_k$ only depend on $k, n, \max\{\alpha, 1\}$, we know

$$\sup_{x\in M} u(x,t) \le \widehat{C}_k\beta^2 \quad \forall t \in \left[0, \frac{\alpha}{\beta}\right], \quad \text{where} \quad \widehat{C}_k = \widehat{C}_k(k,n,\max\{\alpha,1\}).$$

Therefore,

$$|\nabla^k \mathrm{Rm}| \le \sqrt{\frac{u}{t_m}} \le \frac{\widehat{C}_k^{1/2} K}{t^{m/2}} \quad \forall t \in \left(0, \frac{\alpha}{\beta}\right].$$

Setting $C_k(k,n,\max\{\alpha,1\}) = \widehat{C}_k^{1/2}$, we get the conclusion. □

Clearly, BBS estimates are completely useless at $t = 0$, since bounds on an arbitrary curvature tensor will not tell us anything about its derivatives. It is only after a period of Ricci flowing that the derivatives start to be brought under control.

For the sake of convenience, we write down the following trivial corollary.

Corollary 4.6. *Let $g(t)$, $t \in [0,\tau]$, be a solution to the Ricci flow on a closed manifold M^n satisfying*

$$\sup_M |\mathrm{Rm}_{g(t)}| \le \tau^{-1}, \quad \forall t \in [0,\tau]. \tag{4.7}$$

Given any integer $m \ge 1$, there exists a positive constant $C = C(m,n)$ such that

$$\sup_M |\nabla^m \mathrm{Rm}_{g(t)}|^2 \le C\tau^{-m-2}, \quad \forall t \in [\tau/2, \tau].$$

Remark 4.7. Condition (4.7) is easy to meet by choosing the initial time appropriately.

4.D. Cuvature explodes at finite-time singularities.

Theorem 4.8. *If g_0 is a metric on a closed manifold M, the Ricci flow with $g(0) = g_0$ has a unique solution $g(t)$ on a maximal time interval $t \in [0,T)$ where $T \le \infty$. If $T < \infty$ then*

$$\lim_{t\to T}\left(\sup_M |\mathrm{Rm}(x,t)|\right) = \infty. \tag{4.8}$$

Remark 4.9. It suffices to show that if $|\mathrm{Rm}|_g$ is bounded above near T, then $g(t)$ will converge smoothly to a smooth metric $g(T)$, and we can use the short-time existence result (corollary 3.13), with initial metric $g(T)$, to extend the solution past T.

So the key point is to show the convergence of $g(t)$. Our idea is to apply theorem 8.4 based on corollary 4.6.

Proof. Suppose for contradiction that (4.8) is false. Then corollary 4.6 yields that

$$\sup_{t\in[0,T)} \sup_M |\nabla^m \mathrm{Rm}_{g(t)}| < \infty \quad \forall m \in \mathbb{N}.$$

Then by the Ricci flow equation (2.1) and theorem 8.4, we know the metrics $g(t)$ converge in C^∞ to some limit metric $\overline{g}$ on M. Then corollary 3.13 implies that we can extend the solution beyond T; a contradiction. □

4.E. **Derivative estimates for tensors.** In analogue with BBS estimates, one can easily derive the following results based on the scalar maximum principle 8.1.

Theorem 4.10. *Let M be a closed manifold of dimension n, let $g(t)$, $0 \le t \le \tau$, be a solution to the Ricci flow on M satisfying*

$$\sup_M |\mathrm{Rm}_{g(t)}| \le \tau^{-1} \quad \forall t \in [0,\tau], \tag{4.9}$$

and let H be a smooth tensor field satisfying

$$\frac{\partial}{\partial t} H = \Delta H + R * H$$

and

$$\sup_M |H| \le \Lambda \quad \forall t \in [0,\tau].$$

Then we can find a positive constant $C = C(n)$ such that

$$\sup_M |\nabla H|^2 \le C\Lambda^2\tau^{-1} \quad \forall t \in [\tau/2,\tau].$$

Remark 4.11. Condition (4.9) is easy to meet by choosing the initial time appropriately.

Proof. Note that

$$\begin{aligned}\frac{\partial}{\partial t}|\mathrm{Rm}|^2 &= \Delta|\mathrm{Rm}|^2 - 2|\nabla\mathrm{Rm}|^2 + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} \\ &\le \Delta|\mathrm{Rm}|^2 - 2|\nabla\mathrm{Rm}|^2 + C_1\tau^{-3},\end{aligned}$$

where $C_1 = C_1(n)$, and that

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla\mathrm{Rm}|^2 &= \Delta|\nabla\mathrm{Rm}|^2 - 2|\nabla^2\mathrm{Rm}|^2 + \mathrm{Rm} * \nabla\mathrm{Rm} * \nabla\mathrm{Rm} \\ &\le \Delta|\nabla\mathrm{Rm}|^2 + C_2\tau^{-1}\cdot|\nabla\mathrm{Rm}|^2,\end{aligned}$$

where $C_2 = C_2(n)$. Setting $v(t) = t|\nabla\mathrm{Rm}|^2_{g(t)}$, then

$$\frac{\partial}{\partial t}v(t) \le \Delta v(t) + \left(1 + C_2\tau^{-1}t\right)|\nabla\mathrm{Rm}|^2 \le \Delta v(t) + (1 + C_2)\,|\nabla\mathrm{Rm}|^2$$

Setting

$$u(t) = v(t) + C_3|\mathrm{Rm}|^2_{g(t)}, \quad \text{where} \quad C_3(n) = \frac{C_2(n)+1}{2},$$

then

$$\frac{\partial}{\partial t}u(t) \le \Delta u(t) + C_1C_3\tau^{-3},$$

and hence by scalar maximum principle 8.1 we know

$$u(p,t) \le C_1C_3\tau^{-3}t + \sup_{p\in M} u(p,0) \le (C_1C_3 + C_3)\tau^{-2}.$$

Therefore, setting $C_4(n) = C_1(n)C_3(n) + C_3(n)$, we have

$$|\nabla\mathrm{Rm}|^2_{g(t)} \le C_4\tau^{-2}t^{-1}, \quad t \in [0,\tau].$$

On the other hand, note that

$$\begin{aligned}\frac{\partial}{\partial t}|H|^2 &= \Delta|H|^2 - 2|\nabla H|^2 + \mathrm{Rm} * H * H \\ &\leq \Delta|H|^2 - 2|\nabla H|^2 + C_5\tau^{-1}\Lambda^2,\end{aligned}$$

where $C_5 = C_5(n)$, and that

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 &= \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + \mathrm{Rm} * \nabla H * \nabla H + \nabla\mathrm{Rm} * H * \nabla H \\ &\leq \Delta|\nabla H|^2 + C_6\tau^{-1}|\nabla H|^2 + 2C_7C_4^{1/2}\tau^{-1}t^{-1/2}\Lambda|\nabla H| \\ &\leq \Delta|\nabla H|^2 + \left(C_6\tau^{-1}+\tau^{-1}\right)|\nabla H|^2 + C_7^2C_4\tau^{-1}t^{-1}\Lambda^2,\end{aligned}$$

where $C_6 = C_6(n)$, and $C_7 = C_7(n)$. Setting $\widetilde{v}(t) = t|\nabla H|^2_{g(t)}$, then

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{v}(t) &\leq \Delta\widetilde{v}(t) + \left(1 + C_6\tau^{-1}t + \tau^{-1}t\right)|\nabla H|^2 + C_7^2C_4\tau^{-1}\Lambda^2 \\ &\leq \Delta\widetilde{v}(t) + (2 + C_6)|\nabla H|^2 + C_7^2C_4\tau^{-1}\Lambda^2.\end{aligned}$$

Setting

$$\widetilde{u}(t) = \widetilde{v}(t) + C_8|H|^2_{g(t)} \quad \text{where} \quad C_8(n) = \frac{2 + C_6(n)}{2},$$

then

$$\frac{\partial}{\partial t}\widetilde{u}(t) \leq \Delta\widetilde{u}(t) + C_9\tau^{-1}\Lambda^2 \quad \text{where} \quad C_9(n) = C_7^2(n)C_4(n) + C_5(n)C_8(n),$$

and hence by scalar maximum principle 8.1 we know

$$\widetilde{u}(p,t) \leq C_9\tau^{-1}\Lambda^2 t + \sup_{p\in M}\widetilde{u}(p,0) \leq (C_8 + C_9)\Lambda^2.$$

Therefore, setting $C_{10}(n) = C_8(n) + C_9(n)$, we have

$$|\nabla H|^2_{g(t)} \leq C_{10}\Lambda^2 t^{-1}, \qquad t \in [0,\tau].$$

Then the assertion follows. □

5. Hamilton's maximum principle

5.A. **Setting the scene for the maximal principle — the Uhlenbeck trick.** By a standard computation, under the Ricci flow one has

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= (\Delta R)_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}\right) \\ &\quad - \left(R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}\right)\end{aligned} \tag{5.1}$$

where

$$B_{ijkl} = -R_{pij}{}^{q} R_{qlk}{}^{p}. \tag{5.2}$$

However, we can not regard $\mathrm{Rm}(t)$ as a section of some fixed bundle with some fixed metric for each t, which is a basic requirement for applying the maximum principle.

To avoid this, we turn to regarding Rm as a section of $\otimes^4 E^*$, where

$$E = \pi^*(TM), \quad \pi : M \times [0,T) \to M \text{ is the projection.}$$

and E is equipped with the metric h induced by $g(t)$, $0 \le t < T$; namely,

$$X_{(p,t)} \in E_{(p,t)} \implies X_{(p,t)} \in T_pM \quad \text{and} \quad h\left(X_{(p,t)}, Y_{(p,t)}\right) = g_t\left(X_{(p,t)}, Y_{(p,t)}\right).$$

Moreover, since each $X \in \Gamma(E)$ induces $X(t) \in \Gamma(TM)$ by $X(t)(p) = X_{(t,p)}$, we can equip E with a natural connection[6]

$$\begin{aligned} D : \Gamma(T\overline{M}) \times \Gamma(E) &\to \Gamma(E) \\ (X,Y) &\mapsto \nabla_X Y \quad \text{where} \quad (\nabla_X Y)_{(p,t)} = \nabla^{g(t)}_{X_p} Y(t) \\ \left(\frac{\partial}{\partial t}, X\right) &\mapsto \frac{\partial}{\partial t}X - \sum_{k=1}^n \mathrm{Ric}(X, e_k)e_k. \end{aligned}$$

where (e_k) is an orthonormal basis for some T_pM.

Therefore, for a Ricci flow $g(t)$, we can regard the family of Riemannian cuvature as a section $\mathrm{Rm} \in \Gamma(\otimes^4 E^*)$, where $\otimes^4 E^*$ is equipped with the metric and connection induced by h and D.

Proposition 5.1. *The connection D is compatible with the metric h.*

Proof. By definition, clearly it suffices to show that

$$\left(D_{\partial_t} h\right)(X,Y) = 0, \quad \forall X, Y \in \Gamma(E).$$

Note that

$$\begin{aligned} \left(D_{\partial_t} h\right)(X,Y) &= \frac{\partial}{\partial t}\left(h(X,Y)\right) - h\left(D_{\partial_t}X, Y\right) - h\left(X, D_{\partial_t}Y\right) \\ &= \frac{\partial g_t}{\partial t}(X,Y) + g\left(\frac{\partial}{\partial t}X, Y\right) + g\left(X, \frac{\partial}{\partial t}Y\right) \\ &\quad - g\left(\frac{\partial}{\partial t}X - \sum_{k=1}^n \mathrm{Ric}(X,e_k)e_k, Y\right) - g\left(X, \frac{\partial}{\partial t}Y - \sum_{k=1}^n \mathrm{Ric}(Y,e_k)e_k\right) \\ &= \left(\frac{\partial g_t}{\partial t} + 2\mathrm{Ric}\right)(X,Y) = 0. \end{aligned}$$

[6] Here $\overline{M} = M \times [0,T)$ and we use the canonical isomorphism $T\overline{M} \cong TM \oplus \mathrm{span}_{\mathbb{R}}\{\partial_t\}$.

We are done. □

Proposition 5.2 (Uhlenbeck's trick)**.** *For* $\mathrm{Rm} \in \Gamma(\otimes^4 E^*)$ *we have*

$$D_{\partial_t}\mathrm{Rm} = \Delta\mathrm{Rm} + Q(\mathrm{Rm}) \tag{5.3}$$

where $\Delta\mathrm{Rm} = (D^2\mathrm{Rm})(e_i, e_i)$*,*

$$\begin{aligned} Q(\mathrm{Rm})(X,Y,Z,W) &= \mathrm{Rm}(X,Y,e_i,e_j)\mathrm{Rm}(Z,W,e_i,e_j) \\ &\quad +2\mathrm{Rm}(X,e_i,Z,e_j)\mathrm{Rm}(Y,e_i,W,e_j) \\ &\quad -2\mathrm{Rm}(X,e_i,W,e_j)\mathrm{Rm}(Y,e_i,Z,e_j) \end{aligned}$$

and (e_k) *is an orthonormal basis for some* T_pM*.*

Proof. One can refer to [2, proposition 2.14]. □

5.B. **ODE-invariant set.** In this subsection we will give a necessary and sufficient condition for a set to be invariant under an ODE, which is a key part of the maximum principle.

Theorem 5.3. *Let* X *be a finite-dimensional inner product space, let* $\Phi : X \to X$ *be a smooth map, and let* F *be a closed subset of* X*. Then the following statements are equivalent:*

(1) The set F *is invariant under the ODE*[7]

$$\frac{d}{dt}x(t) = \Phi(x(t)). \tag{5.4}$$

(2) It holds that[8]

$$\langle \Phi(y), y - z\rangle \geq 0 \quad \forall z \in X, \quad \forall y \in \mathrm{Proj}_F(z). \tag{5.5}$$

Remark 5.4. It easy to see (1) $\Longrightarrow$ (2).

To see (2) $\Longrightarrow$ (1), the key point is to show that if there exists a solution $x(t)$ that destroys the ODE-invariance of F, then there exist two bounded sequence $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ such that

$$|\Phi(x_k) - \Phi(y_k)| \geq k|x_k - y_k|,$$

which destroys the Lipschitz continuity of Φ.

The idea is as follows. It is the Lipschitz continuity of Φ that ensures the local-in-time existence of $x(t)$, so instead of analyzing the invariant property of solution $x(t)$ via ODE (5.4), which is hard, we turn to showing that a solution without the invariant property doesn't exist, or equivalently, showing that such a solution will destroy the key point of local-in-time existence, the Lipschitz continuity of Φ.

Proof. (1) $\Longrightarrow$ (2): Fix y and z with $y \in \mathrm{Proj}_F(z)$. Let $x(t)$, $0 \leq t < T$, solves ODE (5.4) with $x(0) = y$. By hypothesis, $x(t) \in F$, for all $0 \leq t < T$. Therefore,

$$\begin{aligned} x(0) = y \in \mathrm{Proj}_F(z) &\Longrightarrow |x(t) - z| \geq |x(0) - z|, \quad \forall 0 \leq t < T \\ &\Longrightarrow \frac{d}{dt}\Big|_{t=0}|x(t)-z|^2 \geq 0 \Longrightarrow \langle x'(0), x(0) - z\rangle \geq 0 \\ &\Longrightarrow \langle \Phi(y), y - z\rangle \geq 0. \end{aligned}$$

[7] That is, whenever $x(t)$, $0 \leq t < T$, solves ODE (5.4) with $x(0) \in F$, we have $x(t) \in F$ for all $0 \leq t < T$.

[8] $\mathrm{Proj}_F(z) = \{y \in F : d(z,F) = |y - z|\}$. Since F is closed, this set is never empty.

(2) $\implies$ (1): Suppose that $x(t)$, $0 \le t < T$, solves ODE (5.4) with $x(0) = y$ and $x(\tau) \notin F$ for some $\tau \in (0, T)$.

Claim 5.5. *If $x_k = x(t_k) \notin F$ and $y_k \in \mathrm{Proj}_F(x_k)$, and if*

$$e^{-kt}|x(t) - y_k| \ge e^{-kt_k}|x(t_k) - y_k|, \quad t_k \le t \le \tau,$$

then

$$|\Phi(x_k) - \Phi(y_k)| \ge k|x_k - y_k|.$$

Proof of Claim 5.5. By hypothesis we know

$$\begin{aligned} & e^{-2kt}|x(t) - y_k|^2 \ge e^{-2kt_k}|x(t_k) - y_k|^2 \quad t_k \le t \le \tau \\ \implies & \frac{d}{dt}\Big|_{t=t_k} e^{-2kt}|x(t) - y_k|^2 \ge 0 \implies \langle x'(t_k), x(t_k) - y_k\rangle \ge k|x_k - y_k|^2 \\ \implies & \langle \Phi(x_k) - \Phi(y_k), x_k - y_k\rangle \ge k|x_k - y_k|^2 \implies |\Phi(x_k) - \Phi(y_k)| \ge k|x_k - y_k|. \end{aligned}$$

We are done. □

Now we define t_k by

$$t_k = \sup\left\{t \in [0, \tau] : d(x(t), F) \le e^{kt - k^2}\right\}.$$

Set $x_k = x(t_k)$ and choose $y_k \in \mathrm{Proj}_F(x_k)$. Then for k sufficiently large, we have $t_k \in (0, \tau)$, $d(x(t_k), F) = e^{kt_k - k^2} > 0$, and

$$e^{k(t_k - t)}|x(t) - y_k| \ge e^{k(t_k - t)}d(x(t), F) \ge e^{kt_k - kt}e^{kt - k^2} = |x(t_k) - y_k|, \quad t_k \le t \le \tau$$

By claim 5.5 we know

$$|\Phi(x_k) - \Phi(y_k)| \ge k|x_k - y_k|.$$

Moreover, by the choice of (x_k) and (y_k), we know they are bounded.[9] This contradicts the Lipschitz continuity of Φ. □

5.C. Hamilton's maximum principle.

Hamilton's maximum principle is in analogue with the PDE-ODE principle. We have analyzed the ODE-invariance, and now we show the PDE-invariance.

First we clarify the associated ODE.

Definition 5.6. *Let V be a finite-dimensional vector space equipped with an inner product. We denote by $\mathcal{C}_B(V)$ the space of algebraic curvature tensors on V, i.e. the space of multilinear forms $R : V \times V \times V \times V \to \mathbb{R}$ such that*

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Z, W, X, Y) \quad \forall X, Y, Z, W \in V$$

and

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0 \quad \forall X, Y, Z, W \in V.$$

[9] Since x is continuous, $x([0, \tau])$ is compact and hence bounded. By the definition of t_k and y_k we know (y_k) is bounded.

Proposition 5.7. *If $A \in \mathcal{C}_B(V)$, then $Q(A) \in \mathcal{C}_B(V)$, where*

$$\begin{aligned} Q(A)(X,Y,Z,W) &= A(X,Y,e_i,e_j)A(Z,W,e_i,e_j) \\ &\quad + 2A(X,e_i,Z,e_j)A(Y,e_i,W,e_j) \\ &\quad - 2A(X,e_i,W,e_j)A(Y,e_i,Z,e_j) \end{aligned}$$

and (e_k) is an orthonormal basis for some T_pM.

Proof. Trivial. One can refer to [2, proposition 5.7]. □

Definition 5.8 (Hamilton's ODE)**.** *We call the ODE*

$$\frac{d}{ds}A(s) = Q(A(s)) \quad on \quad \mathcal{C}_B(V) \tag{5.6}$$

*the **Hamilton's ODE**.*

Then we derive an appropriate ODE-invariant subset $F \subset \mathcal{C}_B(E) \subset \otimes^4 E^*$, and prove Hamilton's maximum principle via $F_{(p,t)}$.

Lemma 5.9. *Suppose that $F \subset \mathcal{C}_B(\mathbb{R}^n)$ is $O(n)$-invariant and invariant under the Hamilton ODE. For each (p,t), we find a linear isometry from $\mathbb{R}^n$ to $E_{p,t}$, which induces a linear isometry form $\mathcal{C}_B(\mathbb{R}^n)$ to $\mathcal{C}_B(E_{(p,t)})$. Let $F_{(p,t)}$ be the image of F under this linear isometry. Then*

(1) $F_{(p,t)}$ is well-defined; i.e. $F_{(p,t)}$ is independent of the choice of the linear isometry from $\mathbb{R}^n$ to $E_{(p,t)}$;
(2) $F_{(p,t)}$ is invariant under the Hamilton ODE.

Proof. (1): Since $\mathbb{R}^n$ is equipped with the canonical inner product, for any linear isometries ϕ_1, ϕ_2 from $\mathcal{C}_B(\mathbb{R}^n)$ to $\mathcal{C}_B(E_{(p,t)})$, the linear isometry

$$\phi_2^{-1} \circ \phi_1 \,:\, \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(\mathbb{R}^n)$$

is an action induced by some $g \in O(n)$. Then by the $O(n)$-invariance, we know $F_{(p,t)}$ is well-defined.

(2): Let ϕ be a linear isometry from $\mathbb{R}^n$ to $E_{(p,t)}$, let (e_i) be the canonical orthonormal basis of $\mathbb{R}^n$, and let $\widehat{e_i} = \phi(e_i)$. Note that

$$\frac{d}{ds}\left(A(s)(\widehat{e_i},\widehat{e_j},\widehat{e_k},\widehat{e_l})\right) = Q\left(A(s)(\widehat{e_i},\widehat{e_j},\widehat{e_k},\widehat{e_l})\right) \quad \text{where} \quad A(s) \in \mathcal{C}_B(E_{(p,t)})$$

is equivalent to

$$\frac{d}{ds}\left((\phi^*A)(s)(e_i,e_j,e_k,e_l)\right) = Q\left((\phi^*A)(s)(e_i,e_j,e_k,e_l)\right) \quad \text{where} \quad \phi^*A(s) \in \mathcal{C}_B(\mathbb{R}^n),$$

which implies that

$$\frac{d}{ds}A(s) = Q(A(s)) \text{ on } \mathcal{C}_B(E_{(p,t)}) \iff \frac{d}{ds}(\phi^*A)(s) = Q((\phi^*A)(s)) \text{ on } \mathcal{C}_B(\mathbb{R}^n).$$

Therefore, if $A(s)$ solves ODE (5.4) on $\mathcal{C}_B(E_{(p,t)})$, then $(\phi^*A)(s)$ solves ODE (5.4) on $\mathcal{C}_B(\mathbb{R}^n)$, and hence $(\phi^*A)(s) \in F$ for each s, which implies $A(s) \in \phi(F)$ for each s, where we regard the linear isometry from $\mathcal{C}_B(\mathbb{R}^n)$ to $\mathcal{C}_B(E_{(p,t)})$ still by ϕ. We are done. □

Theorem 5.10 (Hamilton)**.** *Assume that $F \subset \mathcal{C}_B(\mathbb{R}^n)$ is closed, convex, $O(n)$-invariant, and invariant under the Hamilton ODE. Suppose that $g(t)$, $t \in [0,T)$ solves the Ricci flow* *on some closed manifold M^n. Then,*

$$R_{(x,0)} \in F_{(x,0)} \quad \forall x \in M \implies R_{(x,t)} \in F_{(x,t)} \quad \forall x \in M \quad \forall t \in [0,T). \tag{5.7}$$

Remark 5.11. In analogue with theorem 5.3, the idea is to show that any Rm that violates (5.7) will destroy the Lipschitz continuity of Q.

Proof. We define

$$u(t) = \sup_{p\in M} d\left(R_{(p,t)}, F_{(p,t)}\right), \quad 0 \le t < T.$$

Then $u(0) = 0$. Suppose for contradiction that $u(\tau) > 0$ for some $\tau \in (0,T)$.

Claim 5.12. *If $R_k = R_{(p_k,t_k)} \notin F_{(p_k,t_k)}$ and $S_k \in \mathrm{Proj}_{F_{(p_k,t_k)}}(R_{(p_k,t_k)})$, and if*

$$\langle -Q(R_k), S_k - R_k\rangle \ge k|S_k - R_k|^2, \tag{5.8}$$

then

$$|Q(S_k) - Q(R_k)| \ge k|S_k - R_k|.$$

Proof of claim 5.12. By lemma 5.9, we know $F_{(p_k,t_k)}$ is invariant under Hamilton's ODE. Then by theorem 5.3, we know

$$\langle Q(S_k), S_k - R_k\rangle \ge 0.$$

By hypothesis, we have

$$\langle Q(S_k) - Q(R_k), S_k - R_k\rangle \ge k|S_k - R_k|^2,$$

and then the conclusion follows by Cauchy inequality. □

Now we define t_k by [10]

$$t_k = \inf\left\{t \in [0,T) : u(t) \ge e^{kt-k^2}\right\}$$

for k sufficiently large. It is easy to see $t_k \in (0,\tau)$ and $u(t_k) = e^{kt_k-k^2} > 0$. Since M is compact, we can choose $p_k \in M$ such that

$$u(t_k) = d\left(R_{(p_k,t_k)}, F_{(p_k,t_k)}\right).$$

Therefore,

$$e^{k(t_k-t)} d\left(R_{(p,t)}, F_{(p,t)}\right) \le e^{k(t_k-t)} u(t) \le e^{kt_k-kt} e^{kt-k^2} = u(t_k), \quad 0 \le t \le t_k, \quad p \in M. \tag{5.9}$$

Set $R_k = R_{(p_k,t_k)}$ and choose $S_k \in \mathrm{Proj}_{F_{(p_k,t_k)}}(R_{(p_k,t_k)})$. Then by the subsequent proposition 5.13, we know

$$\begin{cases} \left\langle (D_{\partial_t} R)_{(p_k,t_k)}, S_k - R_k \right\rangle \le -k|S_k - R_k|^2, \\ \left\langle (D^2_{v,v} R)_{(p_k,t_k)}, S_k - R_k \right\rangle \ge 0. \end{cases}$$

By Uhlenbeck's trick 5.2, we then get (5.8), and hence by claim 5.12 we know

$$|Q(S_k) - Q(R_k)| \ge k|S_k - R_k|.$$

Moreover, by the choice of (R_k) and (S_k), we know they are bounded. This contradicts the Lipschitz continuity of Φ. □

[10] We need to re-define t_k, since the red inequality in (5.9) reverses the direction of inequality.

Proposition 5.13. *Assume that $F \subset \mathcal{C}_B(\mathbb{R}^n)$ is closed, convex, and $O(n)$-invariant. Moreover, let M be a compact manifold of dimension n, and let $g(t)$, $0 \le t < T$, sloves the Ricci flow on M. Suppse that (p_0, t_0) is a point in $M \times (0, T)$ with the property that*

$$e^{\mu(t_0-t)} d\left(R_{(p,t)}, F_{(p,t)}\right) \le d\left(R_{(p_0,t_0)}, F_{(p_0,t_0)}\right), \quad 0 \le t \le t_k, \quad p \in M.$$

Then for any $S \in \mathrm{Proj}_{F_{(p_0,t_0)}}(R_{(p_0,t_0)})$, we have

(1) $\left\langle (D_{\partial_t} R)_{(p_0,t_0)}, S - R_{(p_0,t_0)} \right\rangle \le -\mu |S - R_{(p_0,t_0)}|^2$;
(2) $\left\langle (D^2_{v,v} R)_{(p_0,t_0)}, S - R_{(p_0,t_0)} \right\rangle \ge 0$ for all $v \in T_{p_0} M$.

Remark 5.14. Since we re-define t_k and reverse the deriction of inequality, these two similar conclusions need convexity essentially.

Proof. (1): For all $s \in [0, t_0)$, let $P(s) : \mathcal{C}_B(E_{(p_0,t_0)}) \to \mathcal{C}_B(E_{(p_0,t_0-s)})$ be the parallel transport with respect to D, and set $H(s) = P(s)^{-1} R_{(p_0,t_0-s)} \in \mathcal{C}_B(E_{(p_0,t_0)})$. Then

$$H(0) = R_{(p_0,t_0)} \quad \text{and} \quad H'(0) = -(D_{\partial_t} R)_{(p_0,t_0)}.$$

Moreover, since F is $O(n)$-invariant, by proposition 5.1, we have $P(s) F_{(p_0,t_0)} = F_{(p_0,t_0-s)}$ for all $s \in [0, t_0)$, and hence[11]

$$e^{\mu s} d(H(s), F_{(p_0,t_0)}) = e^{\mu s} d(R_{(p_0,t_0-s)}, F_{(p_0,t_0-s)}) \le d\left(R_{(p_0,t_0)}, F_{(p_0,t_0)}\right) = |S - H(0)|, \quad 0 \le s < t_0.$$

By lemma 8.5, we have

$$0 \le d(H(s), F_{(p_0,t_0)}) |S - H(0)| + \langle H(s) - S, S - H(0) \rangle, \quad 0 \le s < t_0.$$

Therefore,

$$0 \le e^{-\mu s} |S - H(0)|^2 + \langle H(s) - S, S - H(0) \rangle, \quad 0 \le s < t_0.$$

Then the assertion follows by taking right derivative at 0.

(2): For all $s \in \mathbb{R}$, set $\gamma(s) = \exp_{p_0}(sv)$, let $P(s) : \mathcal{C}_B(E_{(p_0,t_0)}) \to \mathcal{C}_B(E_{(\gamma(s),t_0)})$ be the parallel transport along γ, and set $H(s) = P(s)^{-1} R_{(\gamma(s),t_0)} \in \mathcal{C}_B(E_{(p_0,t_0)})$. Then

$$H(0) = R_{(p_0,t_0)} \quad \text{and} \quad H''(0) = (D^2_{v,v} R)_{p_0,t_0}.$$

Moreover, since F is $O(n)$-invariant, by proposition 5.1, we have $P(s) F_{(p_0,t_0)} = F_{(\gamma(s),t_0)}$ for all $s \in \mathbb{R}$, and hence

$$d(H(s), F_{(p_0,t_0)}) = d\left(R_{(\gamma(s),t_0)}, F_{(\gamma(s),t_0)}\right) \le d\left(R_{(p_0,t_0)}, F_{(p_0,t_0)}\right) = |S - H(0)|, \quad s \in \mathbb{R}.$$

By lemma 8.5, we have

$$0 \le d(H(s), F_{(p_0,t_0)}) |S - H(0)| + \langle H(s) - S, S - H(0) \rangle, \quad s \in \mathbb{R}.$$

Therefore,

$$0 \le |S - H(0)|^2 + \langle H(s) - S, S - H(0) \rangle, \quad s \in \mathbb{R}$$

with equality for $s = 0$. Since 0 is a global minimum, we know

$$\langle H''(0), S - H(0) \rangle = \frac{d^2}{ds^2}\Big|_{s=0} \left(|S - H(0)|^2 + \langle H(s) - S, S - H(0) \rangle \right) \ge 0.$$

Then the assertion follows. □

[11]Since we reverse the deriction of inequality, we can not derive $e^{\mu s}|S - H(s)| \le |S - H(0)|$.

6. HAMILTON'S CONVERGENCE CRITERION

In this subsection, we describe a general method for proving convergence results for the Ricci flow.

As we said in the introduction, we now add the initial pinching condition, and use the scalar maximum principle 8.1 to get a pricise pinching result.

6.A. Pinching set, the hypothesis in the remainder of this section.

First we clarify the initial pinching condition.

Definition 6.1. *Suppose that $R \in \mathcal{C}_B(\mathbb{R}^n)$ and $\delta \in (0,1)$. We say that R is* ***strictly [resp. weakly] δ-pinched*** *if $0 < \delta K(\pi_1) < K(\pi_2)$ [resp. $0 \leq \delta K(\pi_1) \leq K(\pi_2)$] for all two-dimensional planes $\pi_1, \pi_2 \subset \mathbb{R}^n$.*

Definition 6.2. *A set $F \subset \mathcal{C}_B(\mathbb{R}^n)$ is called a pinching set if*

(1) F is closed, convex, $O(n)$-invariant, and invariant under the Hamilton ODE;
(2) for each $\delta \in (0,1)$, the set $\{R \in F : R \text{ is not weakly } \delta\text{-pinched}\}$ is bounded in the sense of sectional curvatures.

Example 6.3. To be continued.

In the remainder of this section, let M be a closed manifold of dimension $n \geq 3$, let g_0 be a metric on M with positive scalar curvature, let $g(t), 0 \leq t < T$ be the unique maximal solution to the Ricci flow with initial metric g_0, and let $F \subset \mathcal{C}_B(\mathbb{R}^n)$ be a pinching set such that $R_{(p,0)} \in F_{(p,0)}$ for all $p \in M$. For abbreviation, we define

$$K_{\max}(t) = \sup_{p\in M} K_{\max}(p,t) \quad \text{and} \quad K_{\min}(t) = \inf_{p\in M} K_{\min}(p,t).$$

6.B. Pinching of sectional curvatures.

Proposition 6.4 (Pointwise pinching)**.** *Given any $\delta \in (0,1)$, we can find a positive constant $C = C(\delta)$ such that*

$$K_{\min}(p,t) \geq \delta K_{\max}(p,t) - C, \quad \forall p \in M, \quad \forall t \in [0,T). \tag{6.1}$$

Moreover, we have

$$\limsup_{t\to T} K_{\max}(t) = \infty.$$

Proof. By Hamilton's maximum principle 5.10, we have $R_{(p,t)} \in F_{(p,t)}$ for all $p \in M$ and all $t \in [0,T)$. Since F is a pinching set, the first assertion follows.

Suppose $\limsup_{t\to T} K_{\max}(t) < \infty$. Then $\sup_{t\in[0,T)} K_{\max}(t) < \infty$. Since $K_{\max}(t) > 0$,[12] by (6.1) we know $\inf_{t\in[0,T)} K_{\max}(t) > -\infty$, and hence $\sup_{t\in[0,T)} |\mathrm{Rm}_{g(t)}| < \infty$;[13] a contradiction. Thus we get the second assertion. □

Theorem 6.5 (Global pinching)**.** *We have*

$$\frac{K_{\min}(t)}{K_{\max}(t)} \to 1 \quad \textit{as} \quad t \to T.$$

[12]Using the scalar maximum principle, one can easily show that the minimum of the scalar curvature is increasing in time, and hence $K_{\max}(t) \geq \frac{\inf_M \mathrm{scal}(t)}{n(n-1)} > 0$.

[13]See [10, problem 3.9].

The above theorem can be easily reduced to the following lemma, which can be proved via the scalar maximum principle and the result of global geometry. (From the result of pointwise pinching, it is natural to attempt to show the global pinching via the tools of global geometry.)[14]

Lemma 6.6. *Let t_k be a sequence of times such that*

$$\lim_{k\to\infty} t_k = T \quad \text{and} \quad K_{\max}(t_k) \geq \frac{1}{2} \sup_{t\in[0,t_k]} K_{\max}(t) \quad \forall k.$$

Then

$$\liminf_{k\to\infty} \frac{K_{\min}(t_k)}{K_{\max}(t_k)} \geq 1.$$

Proof. Fix some $\varepsilon > 0$. Note that

$$\begin{aligned}
&\text{proposition 6.4} \implies \sup_M |\mathring{\mathrm{Ric}}_{g(t)}| \leq \varepsilon K_{\max}(t) + C_1(\varepsilon) \quad t \in [0,T) \quad \text{(corollary 8.9)}\\
\implies &\sup_M |\mathring{\mathrm{Ric}}_{g(t)}| \leq 2\varepsilon K_{\max}(t_k) + C_1(\varepsilon) \quad t \in [0,t_k]\\
\implies &\sup_M |D\mathring{\mathrm{Ric}}_{g(t_k)}|^2 \leq C_2(n)K_{\max}(t_k)(2\varepsilon K_{\max}(t_k) + C_1(\varepsilon))^2 \quad \text{(theorem 4.10) (lemma 8.7)}\\
\implies &\sup_M |d\mathrm{scal}_{g(t_k)}|^2 \leq C_3(n)K_{\max}(t_k)(2\varepsilon K_{\max}(t_k) + C_1(\varepsilon))^2 \quad (\star)
\end{aligned}$$

where $C_1(\varepsilon), C_2(n), C_3(n)$ are positive constants. For each k, choose $p_k \in M$ with $K_{\max}(p_k,t_k) = K_{\max}(t_k)$, and set

$$\Omega_k := B_{r_k}(p_k) \quad \text{where} \quad r_k = 2\pi K_{\max}(t_k)^{-1/2}.$$

Then

$$\begin{aligned}
(\star) \implies &\inf_{x\in\Omega_k} \mathrm{scal}_{g(t_k)}(x) \geq \mathrm{scal}_{g(t_k)}(p_k) - 2\pi C_3(n)^{1/2}(2\varepsilon K_{\max}(t_k) + C_1(\varepsilon))\\
\implies &\inf_{x\in\Omega_k} K_{\max}(x,t_k) \geq K_{\min}(p_k,t_k) - \frac{2\pi}{n(n-1)}C_3(n)^{1/2}(2\varepsilon K_{\max}(t_k) + C_1(\varepsilon)).
\end{aligned}$$

By proposition 6.4 again, there exists a positive constant $C_4(\varepsilon) > 0$ with

$$K_{\min}(p,t_k) \geq (1-\varepsilon)K_{\max}(p,t_k) - C_4(\varepsilon) \quad \forall p \in M.$$

It follows from the above facts that

$$\begin{aligned}
\inf_{x\in\Omega_k} K_{\min}(x,t_k) \;\geq\; &(1-\varepsilon)^2 K_{\max}(t_k) - (2-\varepsilon)C_4(\varepsilon)\\
&-\frac{2\pi}{n(n-1)}C_3(n)^{1/2}(1-\varepsilon)(2\varepsilon K_{\max}(t_k) + C_1(\varepsilon)).
\end{aligned}$$

Note that

$$\lim_{k\to\infty} K_{\max}(t_k) \geq \frac{1}{2} \sup_{t\in[0,T)} K_{\max}(t) = \infty,$$

and hence

$$\liminf_{k\to\infty} \frac{\inf_{x\in\Omega_k} K_{\min}(x,t_k)}{K_{\max}(t_k)} \geq (1-\varepsilon)^2 - \frac{4\pi}{n(n-1)}C_3(n)^{1/2}(1-\varepsilon)\varepsilon.$$

[14] In other words, lemma 6.6 is proposed by trying to use the maximum principle and global geometry.

By the arbitrariness of ε we know

$$\liminf_{k\to\infty} \frac{\inf_{x\in\Omega_k} K_{\min}(x,t_k)}{K_{\max}(t_k)} \geq 1.$$

We claim that $\Omega_k = M$ for k sufficiently large; otherwise, for k sufficiently large choosing $x_k \in M$ with $d_{g(t_k)}(p_k, x_k) = r_k$, then by the diameter theorem 8.6 we have

$$\inf_{x\in\Omega_k} K_{\min}(x,t_k) \leq \frac{\pi^2}{\mathrm{diam}(\Omega_k)^2} \leq \frac{\pi^2}{r_k^2} = \frac{1}{4}K_{\max}(t_k) \quad \text{for } k \text{ sufficiently large,}$$

a contradiction. Hence we get the conclusion. □

Proof of theorem 6.5. Suppose for contraction that there exists a sequence (τ_k) such that

$$\lim_{k\to\infty} \tau_k = T \quad \text{and} \quad \liminf_{k\to\infty} \frac{K_{\min}(\tau_k)}{K_{\max}(\tau_k)} < 1.$$

For each k, there exists $t_k \in [0,\tau_k]$ such that $K_{\max}(t_k) = \sup_{t\in[0,\tau_k]} K_{\max}(t)$. Then by $\lim_{t\to T} K_{\max}(t) = \infty$, we know $\lim_{k\to\infty} K_{\max}(t_k) = \infty$ and $\lim_{k\to\infty} t_k = T$. Then by lemma 6.6, we obtain

$$\liminf_{k\to\infty} \frac{K_{\min}(t_k)}{K_{\max}(t_k)} \geq 1.$$

Using the scalar maximum principle, it is easy to show that the minimum of the scalar curvature is increasing in time. Then

$$\begin{aligned} &\inf_{x\in M} \mathrm{scal}_{g(\tau_k)}(x) \geq \inf_{x\in M} \mathrm{scal}_{g(t_k)}(x) \implies \inf_{x\in M} K_{\max}(x,\tau_k) \geq \inf_{x\in M} K_{\min}(x,t_k) \\ \implies\ & K_{\max}(\tau_k) \geq K_{\min}(t_k) \geq \frac{1}{2}K_{\max}(t_k) = \frac{1}{2}\sup_{t\in[0,\tau_k]} K_{\max}(t) \quad \text{for } k \text{ sufficiently large} \\ \implies\ & \liminf_{k\to\infty} \frac{K_{\min}(\tau_k)}{K_{\max}(\tau_k)} \geq 1 \quad \text{(lemma 6.6)}. \end{aligned}$$

This yields a contraction. □

Corollary 6.7. *We have*

$$(T-t)\sup_M \mathrm{scal}_{g(t)} \to \frac{n}{2} \quad \textit{as} \quad t \to T,$$

and

$$(T-t)\inf_M \mathrm{scal}_{g(t)} \to \frac{n}{2} \quad \textit{as} \quad t \to T.$$

Proof. Assume $\varepsilon > 0$. Then

$$\begin{aligned} &\text{theorem 6.5} \implies |\mathrm{Ric}^\circ|^2 \leq \frac{\varepsilon}{n}\mathrm{scal}^2 \quad \text{on} \quad M\times[T-\eta,T) \quad \text{(corollary 8.10)} \\ \implies\ & \frac{\partial}{\partial t}\mathrm{scal} = \Delta\mathrm{scal} + 2|\mathrm{Ric}|^2 \leq \Delta\mathrm{scal} + \frac{2(1+\varepsilon)}{n}\mathrm{scal}^2 \quad \text{on} \quad M\times[T-\eta,T). \end{aligned}$$

By theorem 6.5 we know $\limsup_{t\to T} \mathrm{scal}_{g(t)} = \infty$. Then it follows from scalar maximum principle 8.1 that

$$(T-t)\sup_M \mathrm{scal}_{g(t)} \geq \frac{n}{2(1+\varepsilon)} \qquad t\in[T-\eta,T).$$

By the arbitrariness of ε, we know

$$\liminf_{t\to T}(T-t)\sup_M \mathrm{scal}_{g(t)} \geq \frac{n}{2},$$

and hence by theorem 6.5 we know

$$\liminf_{t\to T}(T-t)\inf_M \mathrm{scal}_{g(t)} \geq \frac{n}{2}.$$

On the other hand,

$$\begin{aligned} &\frac{\partial}{\partial t}\mathrm{scal} = \Delta\mathrm{scal} + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{n}R^2 \\ \Longrightarrow\quad &(T-t)\inf_M \mathrm{scal}_{g(t)} \leq \frac{n}{2} \quad t\in[0,T) \quad \text{(scalar maximum principle 8.1)} \\ \Longrightarrow\quad &\limsup_{t\to T}(T-t)\inf_M \mathrm{scal}_{g(t)} \leq \frac{n}{2} \\ \Longrightarrow\quad &\limsup_{t\to T}(T-t)\sup_M \mathrm{scal}_{g(t)} \leq \frac{n}{2} \quad \text{(theorem 6.5)}. \end{aligned}$$

We are done. □

6.C. **Bounds on $\omega(t)$.** Now, based on the global pinching result 6.5 and via applying the scalar maximum principle, we get the following bounds.

Lemma 6.8. *Fix $\alpha \in (0, \frac{1}{n-1})$. There exists a positive constant C with*

$$\sup_M |\mathring{\mathrm{Ric}}_{g(t)}|^2 \leq C(T-t)^{2\alpha-2} \quad \forall t\in[0,T),$$

and for each $m\in\mathbb{Z}_+$, there exists a positive constant C_m with

$$\sup_M |\nabla^m \mathring{\mathrm{Ric}}_{g(t)}|^2 \leq C_m(T-t)^{2\alpha-m-2} \quad \forall t\in[0,T).$$

Moreover, there exists a positive constant $\widetilde{C}$ with

$$\sup_M \left|\mathrm{Ric}_{g(t)} - \frac{1}{2(T-t)}g(t)\right|^2 \leq \widetilde{C}(T-t)^{2\alpha-2} \quad \forall t\in[0,T),$$

and for each $m\in\mathbb{Z}_+$, there exists a positive constant $\widetilde{C}_m$ with

$$\sup_M |\nabla^m \mathrm{Ric}_{g(t)}|^2 \leq \widetilde{C}_m(T-t)^{2\alpha-m-2} \quad \forall t\in[0,T).$$

Remark 6.9. For $\widetilde{g}(t) = \frac{1}{2(n-1)(T-t)}g(t)$, we have

$$\omega(t) = \frac{\partial}{\partial t}\widetilde{g}_t = -\frac{1}{(n-1)(T-t)}\left(\mathrm{Ric}_{g(t)} - \frac{1}{2(T-t)}g(t)\right).$$

6.D. **Hamilton's convergence criterion.**

Theorem 6.10 (Hamilton)**.** *let M be a closed manifold of dimension $n\geq 3$, let g_0 be a metric on M with positive scalar curvature, let $g(t)$, $0\leq t<T$ be the unique maximal solution to the Ricci flow with initial metric g_0, and let $F\subset \mathcal{C}_B(\mathbb{R}^n)$ be a pinching set such that $R_{(p,0)}\in F_{(p,0)}$ for all $p\in M$. Then, as $t\to T$, the metrics $\widetilde{g}(t) := \frac{1}{2(n-1)(T-t)}g(t)$ converges uniformly in every C^k norm to a smooth metric $\widetilde{g}(T)$ with constant sectional curvature* 1.

Proof. Set $\omega(t) = \frac{\partial}{\partial t}\widetilde{g}_t$. By lemma 6.8 and theorem 8.4 we know $\widetilde{g}(t)$ converges uniformly in every C^k norm to a smooth metric $\widetilde{g}(T)$. By theorem 6.5, we know $\widetilde{g}(T)$ has constant sectional curvature. By corollary 6.7, the scalar curvature of $\widetilde{g}(T)$ is euqal to $n(n-1)$. This completes the proof. $\square$

7. ODE-INVARIANT CONES, PINCHING CRITERIONS AND THE SPHERE THEOREM

7.A. **Introduction and conventions.** In this section, we aim to establish the 1/4-pinched differentiable sphere theorem.

Theorem 7.1 (S. Brendle, R. Schoen)**.** *Let M be a closed manifold of dimension $n \geq 4$, and let g_0 be a Riemannian metric on M. Assume that (M, g_0) is strictly $1/4$-pinched in the pointwise sense. Let $g(t)$, $t \in [0, T)$, be the unique maximal solution to the Ricci flow with initial metric g_0. Then, as $t \to T$, the metrics $\frac{1}{2(n-1)(T-t)} g(t)$ converges in C^∞ to a metric with constant curvature* 1.

Our tool is Hamilton's convergence theorem 6.10. So our aim is to find the appropriate pinching set. The framework is as follows:

(1) We first give several ODE-invariant cones.
(2) We then give a method to derive pinching sets based on the given cones.
(3) Finally we show g_0 lies in a pinching set.

Conventions through this section:

(1) "ODE-invariance" means the invariance under Hamilton ODE unless we make a clarification.
(2) Given $R \in \mathcal{C}_B(B)$ and some frame (e_i) of V, we write

$$\mathrm{Iso}(R)_{ijkl} = R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} - 2R_{ijkl}$$

and

$$\mathrm{Iso}_{\lambda,\mu}(R)_{ijkl} = R_{ikik} + \lambda^2 R_{ilil} + \mu^2 R_{jkjk} + \lambda^2\mu^2 R_{jljl} - 2\lambda\mu R_{ijkl}.$$

(3) Let X be an inner product space, and let F be a closed and **convex** subset of X. For each $y \in F$ we define

$$N_y F = \{z \in X : \langle x - y, z\rangle \geq 0 \quad \forall x \in F\}$$

and

$$T_y F = \left\{x \in X : \langle x, z\rangle \geq 0 \quad \forall z \in N_y F\right\}.$$

Remark 7.2. (1) If y lies in the interior of F then $N_y F = \{0\}$ and $T_y F = X$.
(2) The property that

$$\langle \Phi(y), y - z\rangle \geq 0 \quad \forall z \in X \quad \forall y \in \mathrm{Proj}_F(z)$$

is equivalent to that $\Phi(y) \in T_y F$ for all points $y \in \partial F$ (or $y \in F$).

7.B. **The cone $\mathcal{C}$.** In this subsection we aim to give a special ODE-invariant cone $\mathcal{C}$ based on lemma 8.11.

Theorem 7.3. *The cone*

$$\mathcal{C} = \{R \in \mathcal{C}_B(\mathbb{R}^n) : R \text{ has non-negative isotropic curvature}\}$$

is closed, convex, $O(n)$-invariant, and ODE-invariant.[15]

[15] An algebraic curvature tensor $R \in \mathcal{C}_B(V)$ is said to have non-negative isotropic curvature if

$$\mathrm{Iso}(R)_{1234} \geq 0 \quad \text{for all orthonormal four-frames } \{e_1, e_2, e_3, e_4\} \subset V.$$

We will use the vital lemma 8.11, for which one can refer to subsection 8.F.

Proof. By definition it is clear that $\mathcal{C}$ is closed, convex and $O(n)$-invariant. In the next we show the ODE-invariance.

The idea is to consider the perturbations and then apply lemma 8.11. Let $R(t)$, $t \in [0, T)$, be a solution to Hamilton ODE with $R(0) \in \mathcal{C}$.

(1) The perturbations is given as follows.
Fix $\varepsilon > 0$, and let $R_\varepsilon(t)$, $t \in [0, T_\varepsilon)$, be the maximal solution to

$$\frac{d}{dt} R_\varepsilon(t) = Q(R_\varepsilon(t)) + \varepsilon I \quad \text{and} \quad R_\varepsilon(0) = R(0) + \varepsilon I \tag{7.1}$$

where $I_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ is the curvature tensor of the standard sphere.

(2) Claim: $R_\varepsilon(t)$ has positive isotropic curvature tensor for $t \in [0, T_\varepsilon)$.
Suppose for contradiction that the claim is false. Setting

$$\tau = \inf \{ t \in [0, T_\varepsilon) : R_\varepsilon(t) \text{ does not have positive isotropic curvature} \},$$

then $\tau \in (0, T_\varepsilon)$. Moreover, there exists an orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ with

$$\mathrm{Iso}(R_\varepsilon(\tau))_{1234} = 0,$$

and hence by lemma 8.11 we know

$$\mathrm{Iso}\,(Q(R_\varepsilon(\tau)))_{1234} \geq 0.$$

However, note that

$$\mathrm{Iso}(R_\varepsilon(t))_{1234} > 0 \quad \forall t \in (0, \tau),$$

and then by the ODE (7.1) we know

$$\mathrm{Iso}\,(Q(R_\varepsilon(\tau)) + \varepsilon I)_{1234} = \mathrm{Iso}\,(Q(R_\varepsilon(\tau)))_{1234} + 4\varepsilon \leq 0.$$

Contradiction.

(3) We obtain $R(t) \in \mathcal{C}$ finally.
The conclusion follows from the standard ODE theory. (One should note that $T \leq \liminf_{\varepsilon \to 0} T_\varepsilon$ and $R(t) = \lim_{\varepsilon \to 0} R_\varepsilon(t)$.)

We are done. □

7.C. The cone $\widehat{\mathcal{C}}$.

Since $\mathcal{C}$ is far away from having non-negative sectional curvature, we need to modify it.

Definition 7.4. *(1) Let V be a vector space of dimension $n \geq 4$ equipped with an inner product. Given any $R \in \mathcal{C}_B(V)$, we define $\widehat{R} \in \mathcal{C}_B(V \times \mathbb{R}^2)$ by*

$$\widehat{R}(\widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{v}_4) = R(v_1, v_2, v_3, v_4)$$

for all vectors $\widehat{v}_j = (v_j, y_j) \in V \times \mathbb{R}^2$.

(2) The modified cone $\widehat{\mathcal{C}}$ is then given by

$$\widehat{\mathcal{C}} = \{ R \in \mathcal{C}_B(\mathbb{R}^n) : \widehat{R} \text{ has non-negative isotropic curvature} \}.$$

Theorem 7.5. *The cone $\widehat{\mathcal{C}}$ is closed, convex, $O(n)$-invariant, and ODE-invariant. Moreover,*

(1) If $R \in \widehat{\mathcal{C}}$, then R has non-negative sectional curvature.

(2) If R has non-negative curvature operator, then $\widehat{R}$ has non-negative isotropic curvature.

Proof. Let $R(t)$, $t \in [0,T)$, be a solution to Hamilton ODE with $\widehat{R}(0) \in \widehat{\mathcal{C}}$. Then the induced curvature tensors $\widehat{R}(t) \in \mathcal{C}_B(\mathbb{R}^n \times \mathbb{R}^2)$ satify $\frac{d}{dt}\widehat{R}(t) = Q(\widehat{R}(t))$, the Hamilton ODE on $\mathcal{C}_B(\mathbb{R}^n \times \mathbb{R}^2)$. Then we know $\widehat{\mathcal{C}}$ is ODE-invariant by theorem 7.3.

(1): Let $R \in \widehat{\mathcal{C}}$, and let $\{e_1, e_2\}$ be an orthonormal two-frame in $\mathbb{R}^n$. We define an orthonormal four-frame $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4\}$ in $\mathbb{R}^n \times \mathbb{R}^2$ by

$$\widehat{e}_1 = (e_1, 0, 0), \quad \widehat{e}_2 = (0, 0, 1), \quad \widehat{e}_3 = (e_2, 0, 0), \quad \widehat{e}_4 = (0, 1, 0).$$

Since $\widehat{R}$ has non-negative isotropic curvature, we have

$$R_{1212} = \mathrm{Iso}(\widehat{R})_{\widehat{1}\widehat{2}\widehat{3}\widehat{4}} \geq 0.$$

(2): Let R be a non-negative algebraic curvature operator on V. Given an orthonormal four-frame $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4\}$ in $V \times \mathbb{R}^2$, write $\widehat{e}_j = (v_j, y_j)$, and define

$$\phi = v_1 \wedge v_3 - v_2 \wedge v_4 \quad \text{and} \quad \psi = v_1 \wedge v_4 + v_2 \wedge v_3.$$

It follows that

$$\mathrm{Iso}(\widehat{R})_{\widehat{1}\widehat{2}\widehat{3}\widehat{4}} = R(\phi, \phi) + R(\psi, \psi) \geq 0.$$

Thus we conclude that $R \in \widehat{\mathcal{C}}$. □

More precisely, we have the following result.

Proposition 7.6. *Let $R \in \mathcal{C}_B(V)$. TFAE:*
(1) $\widehat{R}$ has non-negative isotropic curvature.
(2) $\mathrm{Iso}_{\lambda,\mu}(R)_{1234} \geq 0$, $\forall \lambda, \mu \in [0,1]$ and for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset V$.
(3) $R(\zeta, \eta, \overline{\zeta}, \overline{\eta}) \geq 0$ for all $\zeta, \eta \in V^{\mathbb{C}}$.

Proof. (1) $\Longrightarrow$ (2): Suppose $\widehat{R}$ has non-negative isotropic curvature. Given $\lambda, \mu \in [0,1]$ and an orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, define

$$\widehat{e}_1 = (e_1, (0,0)), \qquad \widehat{e}_2 = (\mu e_2, (0,0)),$$
$$\widehat{e}_3 = \left(e_3, \left(0, \sqrt{1-\mu^2}\right)\right), \qquad \widehat{e}_4 = \left(\lambda e_4, \left(\sqrt{1-\lambda^2}, 0\right)\right),$$

and then $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4\}$ forms an orthonormal frame. By direct computation we get

$$\mathrm{Iso}(\widehat{R})_{\widehat{1}\widehat{2}\widehat{3}\widehat{4}} \geq 0 \implies \mathrm{Iso}_{\lambda,\mu}(R)_{1234} \geq 0.$$

(2) $\Longrightarrow$ (3): Suppose (2) holds. Given $\zeta, \eta \in V^{\mathbb{C}}$, set $\sigma = \mathrm{span}_{\mathbb{C}}\{\zeta, \eta\}$. By proposition 8.12 we find an orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset V$ and $\lambda, \mu \in [0,1]$ such that

$$z =: e_1 + i\mu e_2 \in \sigma \quad \text{and} \quad w =: e_3 + i\lambda e_4 \in \sigma.$$

Then $\sigma = \mathrm{span}_{\mathbb{C}}\{z, w\}$ and hence

$$R(\zeta, \eta, \overline{\zeta}, \overline{\eta}) = cR(z, w, \overline{z}, \overline{w}) \quad \text{for some } c > 0.$$

By the first Bianchi identity we know

$$R(z, w, \overline{z}, \overline{w}) = \mathrm{Iso}_{\lambda,\mu}(R)_{1234}.$$

Then $R(\zeta, \eta, \overline{\zeta}, \overline{\eta}) = c \cdot \mathrm{Iso}_{\lambda,\mu}(R)_{1234} \geq 0$.

(3) $\Longrightarrow$ (1): Suppose (3) holds. Given an orthonormal four-frame $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3, \widehat{e}_4\}$ in $V \times \mathbb{R}^2$, write $\widehat{e}_j = (v_j, y_j)$, and define $\zeta = v_1 + iv_2$ and $\zeta = v_3 + iv_4$. It follows from the

first Bianchi identity that

$$0 \le R(\zeta, \eta, \overline{\zeta}, \overline{\eta}) = \mathrm{Iso}(R)_{1234},$$

and hence $\mathrm{Iso}(\widehat{R})_{\widehat{1}\widehat{2}\widehat{3}\widehat{4}} = \mathrm{Iso}(R)_{1234} \ge 0$. □

Based on this proposition we can also give a new proof of theorem 7.5.

Another proof of theorem 7.5 (1)(2). Property (1) follows from proposition 7.6 directly. Given an orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset V$ and $\lambda, \mu \in [0,1]$, define

$$\phi = e_1 \wedge e_3 - \lambda\mu e_2 \wedge e_4 \quad \text{and} \quad \psi = \lambda e_1 \wedge e_4 + \mu e_2 \wedge e_3,$$

and then

$$R(\phi, \phi) + R(\psi, \psi) \ge 0 \implies \mathrm{Iso}_{\lambda,\mu}(R)_{1234} \ge 0.$$

Hence we get property (2) by proposition 7.6. □

The following result is an important application of proposition 7.6, which explains why we focus on the cone $\widehat{\mathcal{C}}$ when we consider the differentiable sphere theorem.

Theorem 7.7. *Let (M, g) be a Riemannian manifold of dimension $n \ge 4$. Then:*

(1) If (M, g) is weakly $1/4$-pinched in the pointwise sense, then the curvature tensor of (M, g) lies in the cone $\widehat{\mathcal{C}}$ for all points $p \in M$.

*(2) If (M, g) is strictly $1/4$-pinched in the pointwise sense, then the curvature tensor of (M, g) lies in the **interior** of the cone $\widehat{\mathcal{C}}$ for all points $p \in M$.*

Proof. (1): Given any orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_pM$, corollary 8.8 yields

$$R_{1234} \le \frac{2}{3}(K_{\max}(p) - K_{\min}(p)) \le 2K_{\min}(p).$$

Then for all $\lambda, \mu \in [0,1]$ we have

$$\mathrm{Iso}_{\lambda,\mu}(R)_{1234} \ge (1 + \lambda^2 + \mu^2 + \lambda^2\mu^2 - 4\lambda\mu)K_{\min}(p) \ge 0.$$

Therefore we get the first assertion by proposition 7.6.

(2): The second assertion follows similarly. □

7.D. **The cone $\widehat{C}(s)$.** We first introduce a technique discovered by C. Böhm and B. Wilking [1], which inspires the idea of finding a pinching set.

Definition 7.8. *(1) If A, B are two symmetric bilinear form on $\mathbb{R}^n$, then their Kulkarni-Nomizu product $A \owedge B \in \mathcal{C}_B(\mathbb{R}^n)$ is given by*

$$(A \owedge B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} - A_{jl}B_{il} + A_{jl}B_{ik}.$$

(2) For $a, b \ge 0$, we define a linear map $\ell_{a,b} : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(\mathbb{R}^n)$ by

$$\ell_{a,b}(R) = R + b\mathrm{Ric}^\circ \owedge \mathrm{id} + \frac{a}{n}\mathrm{scal} \cdot \mathrm{id} \owedge \mathrm{id}.$$

Proposition 7.9. *For each $s \ge 0$, we define a cone $\widehat{C}(s) \subset \mathcal{C}_B(\mathbb{R}^n)$ by*

$$\widehat{C}(s) = \left\{ l_{a(s),b(s)}(R) : R \in \widehat{C} \quad \textit{and} \quad \mathrm{Ric} \ge \frac{\delta(s)}{n}\mathrm{scal} \cdot \mathrm{id} \right\}$$

where

$$(2a(s), 2b(s), \delta(s)) = \begin{cases} \left(\dfrac{2s+(n-2)s^2}{1+(n-2)s^2}, 2s, 1-\dfrac{1}{1+(n-2)s^2}\right) & 0 < s \le 1/2, \\ \left(2s, 1, 1-\dfrac{4}{n-2+8s}\right) & s > 1/2. \end{cases}$$

Then $\widehat{C}(0) = \widehat{\mathcal{C}}$, and the cones $\widehat{C}(s)$, $s > 0$, have the following properties:
(1) For each $R \in \partial\widehat{C}(s) \setminus \{0\}$, $Q(R)$ lies in the interior of $T_R\widehat{C}(s)$.[16]
(2) I lies in the interior of $\widehat{C}(s)$.
(3) If $R \in \widehat{C}(s)$ for some $s > 1/2$, then R is weakly $\frac{2s-1}{2s+n-1}$-pinched.

Proof. One can refer to [2]. □

Corollary 7.10. *Let F be a closed, convex, $O(n)$-invariant and ODE-invariant set. Set*

$$\mathcal{I}_F = \left\{ s > 0 \,:\, \exists h > 0 \quad \text{s.t.} \quad \forall R \in F \quad R + hI \in \widehat{C}(s) \right\}.$$

(1) We have $s \in \mathcal{I}_{\widehat{C}(s)}$ for each $s > 0$.
(2) If $\mathcal{I}_F \neq \emptyset$ and $\sup \mathcal{I}_F = \infty$, then F is a pinching set.

7.E. **Some criterions of finding pinching sets.**

Proposition 7.11. *Let K be a compact subset of $\mathcal{C}_B(\mathbb{R}^n)$, and let F be the smallest set containing K which is closed, convex, $O(n)$-invariant and ODE-invariant. If $\mathcal{I}_F \neq \emptyset$, then F is a pinching set.*

By corollary 7.10, it suffices to show $\sup \mathcal{I}_F = \infty$. Then the proposition can be easily reduced to the following lemma.

Lemma 7.12. *Fix a compact interval $[\alpha, \beta] \subset (0, \infty)$. Then there exists a real number $\varepsilon = \varepsilon(\alpha, \beta, n) > 0$ with the following property.*

For any closed and ODE-invariant subset $F \subset \mathcal{C}_B(\mathbb{R}^n)$ satisfying

$$R + hI \in \widehat{C}(s) \quad \forall R \in F \quad \text{for some } s \in [\alpha, \beta] \text{ and some } h > 0,$$

the corresponding set

$$\widehat{F} = \left\{ R \in F \,:\, R + 2hI \in \widehat{C}(s+\varepsilon) \right\}$$

is also ODE-invariant, and satisfies

$$\{R \in F \,:\, \mathrm{scal}(R) \le h\} \subset \widehat{F}. \tag{7.2}$$

Proof. It is easy to find ε such that (7.2) holds. To ensure that $\widehat{F}$ is ODE-invariant, the idea is to apply lemma 8.13.

(1) Find $N = N(\alpha, \beta, n)$ such that for any $R \in \widehat{C}(s)$ with $\mathrm{scal}(R) \ge N$ and $s \in [\alpha, \beta + 1]$, we have $Q(R - 2I)$ lies in the interior of the tangent cone $T_R\widehat{C}(s)$.[17]

[16] This implies that $\widehat{C}(s)$ is ODE-invariant.
[17] If we set $A = \{R \in \widehat{C}(s) \,:\, \mathrm{scal}(R) = 1\}$, then $d(A, \partial\widehat{C}(s)) > 0$ by compactness (using the pinching property of $\widehat{C}(s)$ to show the compactness of A). By continuity, $\frac{R}{N} - \frac{2I}{N} \in \widehat{C}(s) \setminus \{0\}$ for sufficiently large N, and then the existence of N follows from proposition 7.9 (1).

(2) Since $\widehat{C}(s)$ vary continuously in s, we can find $\varepsilon = \varepsilon(\alpha, \beta, n) \in (0, 1]$ such that

$$\begin{aligned} &\left\{R \in \mathcal{C}_B(\mathbb{R}^n) : R + I \in \widehat{C}(s),\ \mathrm{scal}(R) \leq N\right\} \\ &\subset \left\{R \in \mathcal{C}_B(\mathbb{R}^n) : R + 2I \in \widehat{C}(s+\varepsilon)\right\} \end{aligned} \tag{7.3}$$

for all $s \in [\alpha, \beta]$.

We claim that ε is as desired. By scaling it suffices to consider $h = 1$. Clearly (7.3) ensures (7.2). It remains to show the ODE-invariance.

Let $R(t)$, $t \in [0, T)$, solves Hamilton's ODE with $R(0) \in \widehat{F}$. It suffices to show $R(t) + 2I \in \widehat{C}(s+\varepsilon)$. Suppose not; we then define

$$t_0 = \inf\left\{t \in [0, T) : R(t) + 2I \notin \widehat{C}(s+\varepsilon)\right\}.$$

Clearly $R(t_0) + 2I \in \widehat{C}(s+\varepsilon)$.

(1) Suppose $\mathrm{scal}(R(t_0)) \geq N$. By the definition of N, $Q(R(t_0))$ lies in the interior of $T_{R(t_0)+2I}\widehat{C}(s+\varepsilon)$. Then lemma 8.13 clearly yields a contradiction.
(2) Suppose $\mathrm{scal}(R(t_0)) < N$. Then by continuity find $t_1 \in (t_0, T)$ with $\mathrm{scal}(R(t)) \leq N$ for all $t \in [t_0, t_1]$. By formula (7.2) we conclude $R(t) + 2I \in \widehat{C}(s+\varepsilon)$ for all $t \in [t_0, t_1]$, which contradicts the definition of t_0.

We are done. □

Now we can easily prove proposition 7.11.

Proof of proposition 7.11. Choose $s_j \in \mathcal{I}_F$ with $\lim_{j\to\infty} s_j = \sup \mathcal{I}_F$. For each j there exists a real number $h_j > 0$ such that

$$R + h_j I \in \widehat{C}(s_j) \quad \forall R \in F.$$

By proposition 7.9 (2) we can increase h_j. So WLOG we assume that

$$h_j \geq \sup\{\mathrm{scal}(R) : R \in K\}.$$

Clearly, $\sup \mathcal{I}_F = \infty$; otherwise lemma 7.12 gives a contradiction. Then the conclusion follows from corollary 7.10. □

Furthermore, we have the following result.

Theorem 7.13. *Suppose that K is a compact set which is contained in the interior of $\widehat{\mathcal{C}}$. Then there exists a pinching set F such that $K \subset F$.*

Proof. Let F be the smallest pinching set containing K which is closed, convex, $O(n)$-invariant and ODE-invariant. Since $\widehat{C}(t)$ vary continuously and $\widehat{C}(0) = \widehat{\mathcal{C}}$, there exists $s_0 > 0$ such that $K \subset \widehat{C}(s_0)$. By proposition 7.9 (1), we know $\widehat{C}(s_0)$ is closed, convex, $O(n)$-invariant and ODE-invariant, and hence $F \subset \widehat{C}(s_0)$. By corollary 7.10 (1) we know $s_0 \in \mathcal{I}_F$.[18] Then by proposition 7.11 we know F is a pinching set. □

7.F. **The sphere theorem.** Now the differentiable sphere theorem follows from the preceding results.

[18]If $s \in \mathcal{I}_G$ and $F \subset G$, then $s \in \mathcal{I}_F$.

Theorem 7.14. *Let M be a closed manifold of dimension $n \geq 4$, and let g_0 be a Riemannian metric on M. Assume that (M, g_0) is strictly $1/4$-pinched in the pointwise sense. Let $g(t)$, $t \in [0, T)$, be the unique maximal solution to the Ricci flow with initial metric g_0. Then, as $t \to T$, the metrics $\frac{1}{2(n-1)(T-t)} g(t)$ converges in C^∞ to a metric with constant curvature 1.*

Proof. It follows from theorem 7.7, theorem 7.13, and theorem 6.10. □

8. Appendix

8.A. Maximum principles.

Theorem 8.1 (The scalar maximum principle)**.** *Let $g(t)$ be a smooth family of metrics on a closed manifold M. Suppose that $u : M \times [0,T) \to \mathbb{R}$ satisfies*

$$\begin{aligned} \frac{\partial u}{\partial t} &\leq \Delta_{g(t)} u + \langle X(t), \nabla u \rangle_{g(t)} + F(u) \\ u(x,0) &\leq C \quad \forall x \in M, \end{aligned}$$

for some constant C, where $X(t)$ is a smooth family of vector fields and $F : \mathbb{R} \to R$ is locally Lipschitz. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is the solution to the ***associated*** *ODE*

$$\frac{d\phi}{dt} = F(\phi) \quad and \quad \phi(0) = C.$$

Then

$$u(x,t) \leq \phi(t)$$

for all $x \in M$ and $t \in [0,T)$ such that $\phi(t)$ exists.

Proof. Setting $v = u - \phi$, then

$$\frac{\partial v}{\partial t} \leq \Delta v + \langle X, \nabla v \rangle + (F(u) - F(\phi)).$$

Fixing $\tau \in (0,T)$, there exist $C_1 = C_1(\tau)$ and $C = C(\tau)$ such that

$$\sup_{M\times[0,\tau]} |u(x,t)| \leq C_1, \quad \sup_{[0,\tau]} |\phi(t)| \leq C_1,$$
$$|F(x) - F(y)| \leq C|x-y| \quad \forall x,y \in [-C_1, C_1].$$

Then

$$\frac{\partial v}{\partial t} \leq \Delta v + \langle X, \nabla v \rangle + C|v| \quad \forall t \in [0,\tau] \quad \text{and} \quad v(0) \leq 0. \tag{8.1}$$

For any $\varepsilon > 0$, we set

$$w_\varepsilon = e^{-Ct} v - \varepsilon(1+t).$$

Clearly it suffices to prove that

$$w_\varepsilon < 0 \quad \text{on} \quad M \times [0,\tau] \quad \forall \varepsilon > 0.$$

Suppose for contradiction that $\{w_\varepsilon = 0\} \neq \emptyset$.[19] Note that $\{w_\varepsilon = 0\}$ is a closed subset of $M \times [0,\tau]$, and hence is a compact set (since M is compact). Since the continuous projection $\pi_2 : M \times [0,\tau] \to [0,\tau]$ maps the compact set $\{w_\varepsilon = 0\}$ to a compact set, we can find (x_0, t_0) such that[20]

$$w_\varepsilon(x_0,t_0) = 0, \quad w_\varepsilon(x,t) = 0 \ \forall t < t_0, \quad \text{and} \quad w_\varepsilon(x,t_0) \leq 0.$$

Clearly $t_0 > 0$; therefore,

$$\frac{\partial w_\varepsilon}{\partial t}(x_0,t_0) \geq 0, \quad \nabla w_\varepsilon(x_0,t_0) = 0, \quad \text{and} \quad \Delta w_\varepsilon(x_0,t_0) \leq 0.$$

[19]Note that $w_\varepsilon(x,0) \leq -\varepsilon$.

[20]t_0 is the first time that w_ε hits 0.

Then one can easily derive a contradiction at point (x_0, t_0) by (8.1). We are done. □

Theorem 8.2 (Tensor maximum principle). *We assume that*

(1) $\pi : E \to M$ *is a vector bundle with a fixed bundle metric* h*;*
(2) $\overline{\nabla}(t)$ *is a smooth family of connections on* E *compatible with* h*;*
(3) $g(t)$ *is a smooth family of metrics on* M*;*
(4) $\mathcal{K}$ *is a subset of* E *that is closed and convex in each fiber, and* $\mathcal{K}$ *is invariant under parallel translation;*
(5) $F : E \times [0, T) \to E$ *is a continuous map that is fiber preserving, and* F *is Lipschitz in each fiber.*

Let $\alpha(t)$ *be a time-dependent section of* E *that satisfies*[21]

$$\frac{\partial}{\partial t}\alpha = \widehat{\Delta}\alpha + F(\alpha), \qquad \alpha(0) \in \Gamma(M, \mathcal{K}). \tag{8.2}$$

If every solution to the ODE

$$\frac{da}{dt} = F(a), \qquad a(0) \in \mathcal{K}_x \tag{8.3}$$

remains in $\mathcal{K}_x$ *where* $\mathcal{K}_x = E_x \cap \mathcal{K}$*, then the solution* $\alpha(t)$ *to the PDE remains in* $\Gamma(M, \mathcal{K})$*.*

Proof. See [3, theorem 4.8]. □

8.B. **Convergence of metrics.**

Theorem 8.3. *Let* $g(t)$ *be a smooth family of metrics on a closed manifold* M*, defined for* $t \in [0, T)$*. If there exists a constant* $C < \infty$ *such that*

$$\int_0^T \left| \frac{\partial}{\partial t} g(x, t) \right|_{g(t)} dt \le C \quad \forall x \in M, \tag{8.4}$$

then the metrics $g(t)$ *converges uniformly as* $t \to T$ *to a continuous metric* $g(T)$ *such that*

$$e^{-C} g(x, 0) \le g(x, T) \le e^{C} g(x, 0). \tag{8.5}$$

Note that this means $g(T)$ *is uniformly equivalent to* $g(0)$*.*

Proof. Clearly (8.4) implies

$$e^{-C} g(x, 0) \le g(x, t) \le e^{C} g(x, 0) \quad \forall t \in [0, T). \tag{8.6}$$

We set

$$g(x, T) = g(x, 0) + \int_0^T \frac{\partial}{\partial t} g(x, t)\, dt.$$

With respect to the norm induced by the fixed metric $g(0)$, using (8.6) one can easily show that $g(t)$ converges to $g(T)$ uniformly on M, and hence $g(T)$ is continuous. Then by taking the limit of (8.6) we get (8.5). □

Theorem 8.4. *Let* $g(t)$*,* $0 \le t < T$*, be a smooth family of metrics on a closed manifold* M*, and let* $\nabla(t)$ *be the Levi-Civita connection of* $g(t)$*. Set*

$$\omega(t) = \frac{\partial}{\partial t} g(t) \quad \textit{and} \quad u_m(t) = \sup_M |\nabla^m \omega(t)|_{g(t)}.$$

[21] We define $\widehat{\nabla}_X(\omega \otimes s) = (\nabla_X \omega) \otimes s + \omega \otimes \overline{\nabla}_X s$, and $\widehat{\Delta}\phi = \mathrm{tr}_g \widehat{\nabla}\overline{\nabla}\phi$.

If

$$\int_0^T u_m(t)\,dt < \infty \quad \forall m = 0, 1, 2, \cdots, \tag{8.7}$$

then $g(t)$ *converges uniformly in every* C^k *norm to a smooth metric* $g(T)$ *as* $t \to T$.

Proof. One can refer to [2, proposition A.5] □

8.C. **Closed and convex subsets of a finite-dimensional inner product space.**

Lemma 8.5. *Let* X *be a finite-dimensional inner product space, and let* F *be a closed, convex subset of* X. *Suppose* $z \in X$ *and* $y \in \mathrm{Proj}_F(z)$. *Then*

$$0 \le d(\tilde{z}, F)|y - z| + \langle \tilde{z} - y, y - z\rangle \quad \forall \tilde{z} \in X.$$

Proof. One can refer to [2, lemma 5.3]. □

8.D. **Global geomtry.**

Theorem 8.6 (Diameter theorem). *Suppose that* (M^n, g) *is a Riemannian n-manifold. If*

$$\mathrm{Ric} \ge (n-1)Kg \quad on \quad B(p, r)$$

for some constant $K > 0$, *then*

$$\mathrm{diam}(B(p, r)) \le \frac{\pi}{\sqrt{K}}.$$

8.E. **Curvature estimates.**

Lemma 8.7. *Let* (M, g) *be a Riemannian manifold. Then the knowledge of all the sectional curvatures determines the curvature tensor. Speaking specifically, setting*

$$\kappa(X, Y) = R(X, Y, Y, X),$$

then we have

$$\begin{aligned} R(X,Y,Z,W) \;=\; & \kappa(X+W, Y+Z) - \kappa(X, Y+Z) - \kappa(W, Y+Z) \\ & -\kappa(Y+W, X+Z) + \kappa(Y, X+Z) + \kappa(W, X+Z) \\ & -\kappa(X+W, Y) + \kappa(X, Y) + \kappa(W, Y) \\ & -\kappa(X+W, Z) + \kappa(X, Z) + \kappa(W, Z) \\ & +\kappa(Y+W, X) - \kappa(Y, X) - \kappa(W, X) \\ & +\kappa(Y+W, Z) - \kappa(Y, Z) - \kappa(W, Z) \end{aligned}$$

Proof. See [6, theorem 6.5]. □

Corollary 8.8. *Given* $R \in \mathcal{C}_B(\mathbb{R}^n)$, *then*

$$R(e_1, e_2, e_3, e_4) \le \frac{2}{3}(K_{max} - K_{\min})$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ *in* R^n.

Proof. It directly follows from lemma 8.7. □

Corollary 8.9. *If given any $\delta \in (0,1)$, we can find a positive constant $C = C(\delta)$ such that*

$$K_{\min}(p,t) \geq \delta K_{\max}(p,t) - C, \quad \forall p \in M, \quad \forall t \in [0,T),$$

then for each $\varepsilon > 0$ we can find a positive constant $C_1 = C_1(\varepsilon)$ such that

$$\sup_M |\overset{\circ}{\mathrm{Ric}}_{g(t)}| \leq \varepsilon K_{\max}(t) + C_1 \quad t \in [0,T)$$

Proof. Fix $\delta \in (0,1)$. Since $K_{\min}|X \wedge Y|^2 \leq \kappa(X,Y) \leq K_{\max}|X \wedge Y|^2$, we know

$$\delta K_{\max}(p,t)|X \wedge Y|^2 - C(\delta)|X \wedge Y|^2 \leq \kappa_{(p,t)}(X,Y) \leq K_{\max}(p,t)|X \wedge Y|^2$$

for all $p \in M$ and $t \in [0,T)$. Then the conclusion easily follows from lemma 8.7. □

Corollary 8.10. *If we have*

$$\lim_{t \uparrow T} \frac{K_{\min}(t)}{K_{\max}(t)} = 1$$

then for each $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that

$$|\overset{\circ}{\mathrm{Ric}}|^2 \leq \frac{\varepsilon}{n}\mathrm{scal}^2 \quad \text{on} \quad M \times [T-\eta, T)$$

Proof. Fix $\delta \in (0,1)$. Then there exists $\alpha = \alpha(\delta) > 0$ such that

$$K_{\min}(t) \geq (1-\delta)K_{\max}(t) \quad \forall t \in [T-\alpha, T)$$

which implies

$$(1-\delta)K_{\max}(t)|X \wedge Y|^2 \leq \kappa_{(p,t)}(X,Y) \leq K_{\max}(t)|X \wedge Y|^2 \quad \forall p \in M \quad \forall t \in [T-\alpha, T).$$

Then by lemma 8.7 one easily know that for each $\delta \in (0,1)$, there exists $\beta = \beta(\delta) > 0$ such that

$$K_{\min}(t) \geq (1-\delta)K_{\max}(t) \quad \forall t \in [T-\beta, T),$$

and that

$$|\overset{\circ}{\mathrm{Ric}}_{g(t)}(p)| \leq \delta K_{\max}(t) \quad \forall p \in M \quad \forall t \in [T-\beta, T).$$

Since

$$\mathrm{scal}_{g(t)}(p) \geq n(n-1)K_{\min}(t)$$

the conclusion follows. □

8.F. **Isotropic curvature.**

Lemma 8.11. *Let R be an algebraic curvature tensor on V with non-negative isotropic curvature, and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal four-frame in V. Then*

$$\mathrm{Iso}(R)_{1234} = 0 \implies \mathrm{Iso}(Q(R))_{1234} \geq 0.$$

8.G. **Results from complex linear algebra.**

Proposition 8.12. *Assume that $\dim_{\mathbb{R}} V \geq 4$. Moreover, suppose that σ is a complex two-plane in $V^{\mathbb{C}}$. Then there exists an orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset V$ and real numbers $\lambda, \mu \in [0,1]$ such that $e_1 + i\mu e_2 \in \sigma$ and $e_3 + i\lambda e_4 \in \sigma$.*

8.H. **Tangent cone.**

Lemma 8.13. *Let X be an inner product space, let F be a closed, convex subset of X, and let $x(t)$, $t \in [0,T)$, be a smooth path in X with $x(0) \in F$. Then*

(1) If $x(t) \in F$ for all $t \in [0,T)$, then $x'(0) \in T_{x(0)}F$.
(2) If $x'(0)$ lies in the interior of the tangent cone $T_{x(0)}F$, then there exists $\varepsilon \in (0,T)$ such that $x(t) \in F$ for all $t \in [0,\varepsilon]$.

9. Appendix — Another approach of Hamilton's maximum principle

9.A. Setting the scene for the maximal principle — the Uhlenbeck trick. By a standard computation, under the Ricci flow one has

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = (\Delta R)_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}\right) \\ - \left(R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}\right)\end{aligned} \tag{9.1}$$

where

$$B_{ijkl} = -R_{pij}{}^{q} R_{qlk}{}^{p}. \tag{9.2}$$

Naively we might try to apply theorem 8.2 to R_{ijkl}. However, we can not treat Δ as $\widehat{\Delta}$, since there is no obvious metric compatible with each $\nabla(t)$. Furthermore, the reaction term are so hideous that the associated ODE is useless.

In the next we introduce the **Uhlenbeck trick**. The idea is as follows.

(1) If we can find a smooth family of bundle isometries
$$\iota(t) : (TM, g_0) \to (TM, g(t)),$$
then each new connection $\overline{\nabla}(t)$ on TM which is given by
$$\overline{\nabla}_j(X) := \iota^{-1}\nabla_j(\iota \circ X)$$
is compatible with g_0.

(2) We should find good $\iota(t)$ such that the evolution equation of $\iota^*\mathrm{Rm}$ is good, which ensures that the associated ODE is easy, and hence we can apply theorem 8.2.

Specifically, one should give $\iota(t)$ by

$$\frac{\partial}{\partial t}\iota = \mathrm{Ric}\circ\iota, \qquad \iota(0) = \mathrm{id}, \tag{9.3}$$

where we regard the Ricci tensor as a $(1,1)$-tensor.

Proposition 9.1. *Formula (9.3) gives a smooth family of bundle isometries.*

Proof. Note that

$$\frac{\partial}{\partial t}\iota = \mathrm{Ric}\circ\iota \iff \frac{\partial}{\partial t}\iota_a^i = R_l^i \iota_a^l. \tag{9.4}$$

It follows that

$$\frac{\partial}{\partial t}\left(\iota(t)^* g(t)\right)_{ab} = \frac{\partial}{\partial t}\iota_a^i \iota_b^j g_{ij} = R_l^i \iota_a^l \iota_b^j g_{ij} + \iota_a^i R_l^i \iota_b^l g_{ij} + \iota_a^i \iota_b^j(-2R_{ij}) = 0.$$

We are done. □

Proposition 9.2. *Each new connection $\overline{\nabla}(t)$ on TM which is given by*

$$\overline{\nabla}_j(X) := \iota^{-1}\nabla_j(\iota \circ X)$$

is compatible with g_0.

Proof. Note that

$$\begin{aligned}
\partial_j \langle X, Y\rangle_{g_0} &= \partial_j \langle \iota(t)\circ X, \iota(t)\circ Y\rangle_{g(t)} \\
&= \left\langle \nabla_j(\iota\circ X), \iota\circ Y\right\rangle_{g(t)} + \left\langle \iota\circ X, \nabla_j(\iota\circ Y)\right\rangle_{g(t)} \\
&= \left\langle \iota\circ\overline{\nabla}_j X, \iota\circ Y\right\rangle_{g(t)} + \left\langle \iota\circ X, \iota\circ\overline{\nabla}_j Y\right\rangle_{g(t)} \\
&= \left\langle \overline{\nabla}_j X, Y\right\rangle_{g_0} + \left\langle X, \overline{\nabla}_j Y\right\rangle_{g_0}.
\end{aligned}$$

We are done. □

Proposition 9.3. *The evolution equation of* $\iota^*\mathrm{Rm}$ *is*

$$\frac{\partial}{\partial t}R_{abcd} = (\widehat{\Delta}R)_{abcd} + 2\left(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}\right) \tag{9.5}$$

where B_{abcd} *is given by (9.2). Equivalently, suppose that* $\{e_k\}$ *is an orthonormal local frame; then the evolution equation can be written as*

$$\frac{\partial}{\partial t}R_{abcd} = \widehat{\Delta}R_{abcd} + R^2_{abcd} + R^\sharp_{abcd} \tag{9.6}$$

where

$$R^2_{abcd} = R_{abef}R_{cdef} \quad \textit{and} \quad R^\sharp_{abcd} = 2R_{aecf}R_{bedf} - 2R_{aedf}R_{becf}.$$

Proof. By formula (9.1) and formula (9.4) one easily computes

$$\frac{\partial}{\partial t}R_{abcd} = \iota^i_a\iota^j_b\iota^k_c\iota^l_d\left[\Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}\right)\right].$$

One can also easily show that $[\iota^*(\Delta\mathrm{Rm})]_{abcd} = (\widehat{\Delta}R)_{abcd}$ and $B_{abcd} = (\iota^*B)(\partial_a, \partial_b, \partial_c, \partial_d)$. Then (9.5) follows. Clearly $(\iota^*\mathrm{Rm})_{abcd} = R_{abcd}$ satisfies the first Bianchi identity

$$R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

Then (9.6) easily follows. □

Remark 9.4. Formula (9.6) simplifies the reaction term, and hence the associated ODE is also simplified.

Up to now we have set the scene for the tensor maximum principle 8.2: We aim to apply it to $\iota^*\mathrm{Rm}$ with respect to $(\otimes^4T^*M, g_0)$. The associated ODE of PDE (9.6) is

$$\frac{d}{dt}Q_{abcd} = Q^2_{abcd} + Q^\sharp_{abcd}. \tag{9.7}$$

where

$$Q^2_{abcd} = Q_{abef}Q_{cdef} \quad \text{and} \quad Q^\sharp_{abcd} = 2Q_{aecf}Q_{bedf} - 2Q_{aedf}Q_{becf}. \tag{9.8}$$

Definition 9.5. *We call the ODE (9.7) the **Hamilton ODE**.*

9.B. **Hamilton's maximum principle for the Ricci flow.** In the next we give a basic principle how we apply the tensor maximum principle to the Ricci flow. The key point is to derive an appropriate set $\mathcal{K}$ that is invariant under the Hamilton ODE.

Definition 9.6. *Let* V *be a finite-dimensional vector space equipped with an inner product. We denote by* $\mathcal{C}_B(V)$ *the space of algebraic curvature tensors on* V*, i.e. the space of* *multilinear forms* $R : V \times V \times V \times V \to \mathbb{R}$ *such that*

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Z,W,X,Y) \quad \forall X,Y,Z,W \in V$$

and

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0 \quad \forall X,Y,Z,W \in V.$$

Proposition 9.7. *If* $Q \in \mathcal{C}_B(V)$*, then* $Q^2 + Q^\sharp \in \mathcal{C}_B(V)$*, where* Q^2 *and* $Q^\sharp$ *are given by (9.8).*

Definition 9.8. *We call a set* $F \subset \mathcal{C}_B(\mathbb{R}^n)$ *is* ***invariant under the Hamilton ODE****, if for any* $Q(t)$ *solving the Hamilton ODE (9.7) on* $\mathcal{C}_B(\mathbb{R}^n)$ *with* $Q(0) \in F$ *we have* $Q(t) \in F$*,* $\forall t$*.*

Lemma 9.9. *Assume that* $F \subset \mathcal{C}_B(\mathbb{R}^n)$ *is closed, convex,* $O(n)$*-invariant and invariant under the Hamilton ODE. For each* $x \in M$*, we find a linear isometry*[22]

$$\Phi_x : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(T_x^*M)$$

and we define

$$\mathcal{K}_x := \Phi_x(F) \subset \mathcal{C}_B(T_x^*M) \subset \otimes^4 T_x^*M.$$

Setting $\mathcal{K} = \cup_x \mathcal{K}_x$*, then*

(1) $\mathcal{K}$ *is independent from the choice of* Φ_x*;*
(2) $\mathcal{K}$ *is a subset of* $\otimes^4 T^*M$ *that is closed and convex in each fiber;*
(3) $\mathcal{K}$ *is invariant under parallel translation;*
(4) $\mathcal{K}_x$ *is invariant under the Hamilton ODE.*

Proof. Since $\mathbb{R}^n$ is equipped with the canonical inner product, the linear isometry

$$\left(\widetilde{\Phi}_x\right)^{-1} \circ \Phi_x : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(\mathbb{R}^n)$$

is an action induced by some $g \in O(n)$. Then point (1) follows from the $O(n)$-invariance. Point (2) is trivial. Since parallel translation keeps the inner product, then point (3) follows from the $O(n)$-invariance. Point (4) holds since Φ_x is a linear isometry. □

Lemma 9.10. *Assume that* $F \subset \mathcal{C}_B(\mathbb{R}^n)$ *is* $O(n)$*-invariant. Suppose that* $g(t)$*,* $t \in [0,T)$ *solves the Ricci flow on some closed manifold* M^n*. For each* $(x,t) \in M \times [0,T)$*, we find a linear isometry*

$$\Psi_{(x,t)} : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(T_x^*M, g(t))$$

and we define

$$F_{(x,t)} = \Psi_{(x,t)}(F) \subset \mathcal{C}_B(T_x^*M, g(t)) \subset \otimes^4(T_xM, g(t)).$$

Then $F_{(x,t)}$ *is independent from the choice of* $\Psi_{(x,t)}$*.*

Proof. Similar to lemma 9.9. □

Theorem 9.11 (Hamilton)**.** *Assume that* $F \subset \mathcal{C}_B(\mathbb{R}^n)$ *is closed, convex,* $O(n)$*-invariant, and invariant under the Hamilton ODE. Suppose that* $g(t)$*,* $t \in [0,T)$ *solves the Ricci flow on some closed manifold* M^n*. Then,*

$$R_{(x,0)} \in F_{(x,0)} \quad \forall x \in M \implies R_{(x,t)} \in F_{(x,t)} \quad \forall x \in M \quad \forall t \in [0,T).$$

[22] $\mathbb{R}^n$ is equipped with the canonical inner product.

Proof. By lemma 9.9 and the tensor maximum principle 8.2,

$$\iota(t)^*\mathrm{Rm}_x \in \mathcal{K}_x \quad \forall x \in M \quad \forall t \in [0, T)$$

where $\iota(t)$ is given by (9.3). Since $\iota(t)$ is a bundle isometry, by lemmas 9.9 and 9.10, WLOG we assume that the following diagram commutes

$$\begin{array}{ccc} & F \subset \mathcal{C}_B(\mathbb{R}^n) & \\ {}^{\Psi_{(x,t)}}\swarrow & & \searrow^{\Phi_x} \\ F_{(x,t)} \subset \mathcal{C}_B(T_xM, g_x(t)) & \xrightarrow[\iota(t)^*]{} & \mathcal{K}_x \subset \mathcal{C}_B(T_xM, g_0) \end{array}$$

where each arrow is an isometry. Then we get the conclusion. □

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