RIEMANNIAN GEOMETRY

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ABSTRACT. To be continued.

CONTENTS

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1. Introduction

To be continued.

2. Basic review of Riemannian geometry

2.A. Basic settings.

Definition 2.1 (Affine connection). *An **affine connection** ∇ on the vector bundle $E \to M$ is a map*

$$\nabla : \Gamma(M, TM) \times \Gamma(M, E) \to \Gamma(M, E), \quad (X, s) \nabla \nabla_X s$$

such that for any $X, Y \in \Gamma(M, TM)$, $s, t \in \Gamma(M, E)$ and $f \in C^\infty(M)$ one has

$$\nabla_{fX+Y} s = f \nabla_X s + \nabla_Y s,$$

and the Leibniz rule

$$\nabla_X (fs + t) = X(f)s + f \nabla_X s + \nabla_X t.$$

Definition 2.2 (Curvature). *The **curvature** R^∇ of the affine connection ∇ is defined as*

$$(R^\nabla s)(X, Y) = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]} s$$

for $X, Y \in \Gamma(M, TM)$ and $s \in \Gamma(M, E)$.

Proposition 2.3. *In local charts (U, ϕ, x^i) of M and local coordinates (U, ψ, e_α) of E,*

$$\nabla_{\frac{\partial}{\partial x^i}} e_\alpha = \Gamma^\beta_{i\alpha} e_\beta,$$

*where $\Gamma^\beta_{i\alpha}$ are called the **Christoffel symbols** of the affine connection ∇. Moreover,*

$$R^\nabla = R^\beta_{ij\alpha} dx^i \otimes dx^j \otimes e^\alpha \otimes e_\beta$$

where

$$R^\beta_{ij\alpha} = \frac{\partial \Gamma^\beta_{j\alpha}}{\partial x^i} - \frac{\partial \Gamma^\beta_{i\alpha}}{\partial x^j} + \Gamma^\gamma_{j\alpha} \Gamma^\beta_{i\gamma} - \Gamma^\gamma_{i\alpha} \Gamma^\beta_{j\gamma}.$$

Theorem 2.4 (Existence of Levi-Civita connection). *Let (M, g) be a smooth Riemannian manifold. There exists a unique affine connection ∇ which satisfies*

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \quad \textit{(metric compatible)};$$

$$\nabla_X Y - \nabla_Y X = [X, Y]. \quad \textit{(torsion free)}$$

*This connection is called the **Levi-Civita connection** of (M, g).*

Proof. It holds that

$$2 \langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle .$$

□

Definition 2.5 (Riemannian curvature tensor). *Let (M, g) be a smooth Riemannian manifold and ∇ be the Levi-Civita connection. The **curvature tensor** $R : \Gamma(M, TM) \times \Gamma(M, TM) \times \Gamma(M, TM) \to \Gamma(M, TM)$ of (M, g, ∇) is defined as*

$$(X, Y, Z) \mapsto R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]} Z.$$

Proposition 2.6. *In local coordinates (U, x^i), for Levi-Civita connection we have*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad \textit{where} \quad \Gamma^k_{ij} = \Gamma^k_{ji} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right).$$

Moreover, we have

$$R_{ijk}{}^{l} = \partial_i \Gamma^l_{kj} + \partial_j \Gamma^l_{ki} - \Gamma^l_{pj}\Gamma^p_{ki} + \Gamma^l_{pi}\Gamma^p_{kj} \quad \textit{and} \quad R_{ijkl} = g_{pl} R_{ijk}{}^{l}.$$

Remark 2.7. Sometimes we write $R^l_{ijk} = R_{ijk}{}^{l}$.

Theorem 2.8 (Properties of Riemannian curvature tensor)**.** *For Riemannian curvature tensor we have the following properties.*

(1) Symmetry and skew-symmetry:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y).$$

(2) The first Bianchi identity:

$$R(\{X,Y,Z\},W) = 0 \quad \textit{where} \quad R(\{X,Y,Z\},W) = R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W).$$

(3) The second Bianchi identity:

$$(\nabla R)(\{X,Y,Z\},W,T) = 0 \quad \textit{where} \quad (\nabla R)(X,Y,Z,W,T) = (\nabla_X R)(Y,Z,W,T)$$

Definition 2.9 (Sectional curvature)**.** *Let (M,g) be a Riemannian manifold. Given a two dimensional subspace $\Pi = \operatorname{span}\{X,Y\} \subset T_pM$, we define by*

$$K(\Pi) = K(X,Y) = \frac{R(X,Y,Y,X)}{\langle X\wedge Y, X\wedge Y\rangle} = \frac{R(X,Y,Y,X)}{|X|^2|Y|^2 - |\langle X,Y\rangle|^2}$$

the ***sectional curvature*** *of* Π.

Remark 2.10. $K(\Pi)$ is independent from the choice of basis.

Theorem 2.11. *Let (M,g) be a Riemannian manifold. Then the knowledge of all the sectional curvatures determines the curvature tensor. Speaking specifically, setting*

$$\kappa(X,Y) = R(X,Y,Y,X),$$

then we have

$$\begin{aligned} R(X,Y,Z,W) \;=\; & \kappa(X+W,Y+Z) - \kappa(X,Y+Z) - \kappa(W,Y+Z) \\ & -\kappa(Y+W,X+Z) + \kappa(Y,X+Z) + \kappa(W,X+Z) \\ & -\kappa(X+W,Y) + \kappa(X,Y) + \kappa(W,Y) \\ & -\kappa(X+W,Z) + \kappa(X,Z) + \kappa(W,Z) \\ & +\kappa(Y+W,X) - \kappa(Y,X) - \kappa(W,X) \\ & +\kappa(Y+W,Z) - \kappa(Y,Z) - \kappa(W,Z). \end{aligned}$$

Definition 2.12 (Ricci curvature and scalar curvature)**.** *The* ***Ricci curvature*** *of (M,g) is defined by*

$$\mathrm{Ric}(g) = R_{jk}\,dx^j \otimes dx^k \quad \textit{where} \quad R_{jk} = g^{il}R_{ijkl}.$$

The ***scalar curvature*** *of (M,g) is*

$$S = \mathrm{tr}_g \mathrm{Ric} = g^{jk}R_{jk}.$$

Remark 2.13. Sometimes we write $S = R =$ scal, and if we write $S = R$ then the Riemannian curvature tensor is denoted by Rm.

Definition 2.14 (Induced connection). *For any vector field $X \in \Gamma(M, TM)$, the Levi-Civita connection ∇ induces a map*

$$\nabla_X : \Gamma(M, \otimes^r T^*M \otimes \otimes^s TM) \to \Gamma(M, \otimes^r T^*M \otimes \otimes^s TM), \quad T \mapsto \nabla_X T.$$

where

$$\begin{aligned}(\nabla_X T)(Y_1, \cdots, Y_r, \omega_1, \cdots, \omega_s) &= X\left(T(Y_1, \cdots, Y_r, \omega_1, \cdots, \omega_s)\right) \\ &\quad - \sum_{i=1}^{r} T\left(Y_1, \cdots, \nabla_X Y_i, \cdots, Y_r, \omega_1, \cdots, \omega_s\right) \\ &\quad - \sum_{j=1}^{s} T\left(Y_1, \cdots, Y_r, \omega_1, \cdots, \nabla_X \omega_j, \cdots, \omega_s\right).\end{aligned}$$

Proposition 2.15. *In a local chart (U, x^i), we have*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = -\Gamma^j_{ik} dx^k.$$

Definition 2.16 (Covariant derivative). *Let $T \in \Gamma(M, \otimes^r T^*M \otimes \otimes^s TM)$. The* ***covariant derivative*** *$\nabla T \in \Gamma(M, \otimes^{r+1} T^*M \otimes \otimes^s TM)$ is defined by $(\nabla T)(X, \cdots) = (\nabla_X T)(\cdots)$.*

Proposition 2.17. *It holds that*

$$\nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = \partial_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r} + \sum_{m=1}^{s} \Gamma^{j_m}_{ip} T^{j_1 \cdots j_{m-1} p j_{m+1} \cdots j_s}_{i_1 \cdots i_r} - \sum_{l=1}^{r} \Gamma^q_{i i_l} T^{j_1 \cdots j_s}_{i_1 \cdots i_{l-1} q i_{l+1} \cdots i_r}.$$

Remark 2.18. Sometimes we write $\nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = T^{j_1 \cdots j_s}_{i_1 \cdots i_r\, ;i}$.

Proposition 2.19. *Let $S \in \Gamma(M, \otimes^r T^*M \otimes \otimes^s TM)$ and $T \in \Gamma(M, \otimes^a T^*M \otimes \otimes^b TM)$ with*

$$\begin{aligned} S &= S^{j_1 \cdots j_s}_{i_1 \cdots i_r} dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}; \\ T &= T^{q_1 \cdots q_b}_{p_1 \cdots p_a} dx^{p_1} \otimes \cdots \otimes dx^{p_a} \otimes \frac{\partial}{\partial x^{p_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{p_a}}. \end{aligned}$$

Set $\theta = S \otimes T$. Then

$$\theta^{j_1 \cdots j_s, q_1 \cdots q_b}_{i_1 \cdots i_r, p_1 \cdots p_a} = S^{j_1 \cdots j_s}_{i_1 \cdots i_r} \cdot T^{q_1 \cdots q_b}_{p_1 \cdots p_a},$$

and we have the ***Leibniz rule****:*

$$\nabla_i \theta^{j_1 \cdots j_s, q_1 \cdots q_b}_{i_1 \cdots i_r, p_1 \cdots p_a} = \nabla_i S^{j_1 \cdots j_s}_{i_1 \cdots i_r} \cdot T^{q_1 \cdots q_b}_{p_1 \cdots p_a} + S^{j_1 \cdots j_s}_{i_1 \cdots i_r} \cdot \nabla_i T^{q_1 \cdots q_b}_{p_1 \cdots p_a}.$$

Theorem 2.20 (Ricci identity for covariant derivatives). *It holds that*

$$\nabla_k \nabla_l T^{j_1 \cdots j_s}_{i_1 \cdots i_r} - \nabla_l \nabla_k T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = \sum_{m=1}^{s} R^{j_m}_{klp} T^{j_1 \cdots j_{m-1} p j_{m+1} \cdots j_s}_{i_1 \cdots i_r} - \sum_{t=1}^{r} R^q_{kl i_t} T^{j_1 \cdots j_s}_{i_1 \cdots i_{l-1} q i_{l+1} \cdots i_r}.$$

In particular,

$$\nabla_k \nabla_l X^i - \nabla_l \nabla_k X^i = R^i_{klp} X^p$$

and

$$\nabla_k \nabla_l \eta_i - \nabla_l \nabla_k \eta_i = -R^s_{kli} \eta_s.$$

Remark 2.21. $\nabla_k \nabla_l T^{j_1 \cdots j_s}_{i_1 \cdots i_r}$ is the component of $\nabla^2 T$. Precisely, one has

$$\nabla_k \nabla_l T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = \left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} T - \nabla_{\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l}} T \right) \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_r}}, dx^{j_1}, \cdots, dx^{j_s} \right).$$

2.B. **Typical computations.**

Example 2.22. See [Xiob] for some typical examples of computations.

To be continued.

3. Pullback bundle, pullback metric and pullback connection

3.A. **Pullback bundles.** Let $f : M \to N$ be a smooth map, and let $E \xrightarrow{\pi} N$ be a smooth vector bundle. Then

$$f^*E = \{(p, u_E) \in M \times E : f(p) = \pi(u_E)\} \subset M \times E$$

with $\pi_f(p, u_E) = p$ fomrs a smooth vector bundle over M, and f^*E is an embedded submanifold of $M \times E$. For more basic properties of pullback bundles such as the universal property, one can refer to [Xioa].

3.B. **Pullback metrics and pullback connections.** For any (local) section e of E, its induced (local) section of f^*E is given by

$$\widehat{e}(x) = (f^*e)(x) := (x, e(f(x))).$$

Clearly, if (e_A) is a local frame of E, then $(\widehat{e}_A)$ is a local frame of f^*E. Let (x^i) and (y^α) be local coordinates of M and N respectively.

(1) If there is a metric g on E, then the **pullback metric** $\widehat{g}$ on f^*E is $\widehat{g} = f^*g$. That is,

$$\widehat{g}(\widehat{e}_A, \widehat{e}_B)(x) = g(e_A, e_B)(f(x)).$$

(2) If the affine connection ∇ on E is given by

$$\nabla e_A = \Gamma^B_{\alpha A} dy^\alpha \otimes e_B,$$

then the **pullback connection** $\widehat{\nabla}$ on f^*E is given by

$$\widehat{\nabla}\widehat{e}_A = f^*(\nabla e_A) := f^*\left(\Gamma^B_{\alpha A} dy^\alpha\right) \otimes f^*e_B = \frac{\partial f^\alpha}{\partial x^i} \cdot \Gamma^B_{\alpha A} \circ f \cdot dx^i \otimes \widehat{e}_B. \tag{3.1}$$

Remark 3.1. For convenience, we also set $f^*_x w = (x, w) \in f^*E \subset M \times E$ for $w \in E_{f(x)}$.

Proposition 3.2. *The pullback connection $\widehat{\nabla}$ is an affine connection.*

Proof. More precisely, the pullback connection $\widehat{\nabla}$ is defined in the following two steps:

(1) Set $\widehat{\nabla}\widehat{e} = f^*(\nabla e)$ for any section e of E.
(2) Set $\widehat{\nabla}(h\widehat{e}_1 + \widehat{e}_2) = dh \otimes \widehat{e}_1 + \widehat{\nabla}\widehat{e}_2$ for all $h \in C^\infty(M)$ and sections e_1, e_2 of E.

To show that $\widehat{\nabla}$ is a well-defined affine connection, we need to show that the expression $f^*(\nabla e)$ is globally defined, and that (1) and (2) are compatible.

On the one hand, note that

$$\left(f^*(dy^\alpha) \otimes f^*\left(\nabla_{\frac{\partial}{\partial y^\alpha}} e\right)\right)(x) = f^*_x\left(dy^\alpha\big|_{f(x)}\right) \otimes \left(x, \nabla_{\frac{\partial}{\partial y^\alpha}\big|_{f(x)}} e\right).$$

It follows that

$$f^*(dy^\alpha) \otimes f^*\left(\nabla_{\frac{\partial}{\partial y^\alpha}} e\right)$$

is independent of the choice of (y^i). Hence $f^*(\nabla e)$ is globally defined.

On the other hand, clearly we have $\widehat{he} = h\circ f \cdot \widehat{e}$, and to show that (1) and (2) are compatible we need to show that

$$f^*(\nabla(he)) = d(h\circ f) \otimes \widehat{e} + h\circ f \cdot \widehat{\nabla}\widehat{e}.$$

Note that

$$f^*(h\nabla e) = h\circ f \cdot f^*(\nabla e) = h\circ f \cdot \widehat{\nabla}\widehat{e}$$

and hence

$$f^*(\nabla(he)) = f^*(dh \otimes e + h\nabla e) = d(h\circ f) \otimes \widehat{e} + h\circ f \cdot \widehat{\nabla}\widehat{e}.$$

We are done. □

Corollary 3.3. *In general, we have*

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}}(h^A\widehat{e}_A) = \frac{\partial h^A}{\partial x^i} \cdot \widehat{e}_A + h^A \frac{\partial f^\alpha}{\partial x^i} \cdot \Gamma^B_{\alpha A}\circ f \cdot \widehat{e}_B. \tag{3.2}$$

Proof. It follows from proposition 3.2 and formula (3.1). □

Proposition 3.4. *There are some basic formulas about pullback connections.*

(1) $(f^*e)(x) = f^*_x(e(f(x)))$ *for any (local) section e of E.*

(2) $f^*(h\cdot e) = h\circ f \cdot f^*e$, *for any* $h \in C^\infty(N)$ *and for any (local) section e of E.*

(3) $f^*_x(c\cdot w) = c \cdot f^*_x w$, *for any* $w \in E_{f(x)}$ *and for any* $c \in \mathbb{R}$.

(4) For any (local) section e of E, and for any $v \in T_xM$, *there holds*

$$\widehat{\nabla}_v\widehat{e} = f^*_x(\nabla_{f_*v}e). \tag{3.3}$$

Proof. Claims (1) (2) (3) are trivial. For (4) we note that

$$\begin{aligned}
\widehat{\nabla}_v\widehat{e} &= (f^*(\nabla e))(v) = \left(f^*\left(dy^\alpha \otimes \nabla_{\frac{\partial}{\partial y^\alpha}} e\right)\right)(v) \\
&= df^\alpha(v) \cdot f^*_x\left(\nabla_{\frac{\partial}{\partial y^\alpha}\big|_{f(x)}} e\right) = f^*_x\left(df^\alpha(v) \cdot \nabla_{\frac{\partial}{\partial y^\alpha}\big|_{f(x)}} e\right) \\
&= f^*_x\left(\nabla_{dy^\alpha(f_*v)\frac{\partial}{\partial y^\alpha}\big|_{f(x)}} e\right) = f^*_x(\nabla_{f_*v}e).
\end{aligned}$$

where $f^\alpha = y^\alpha\circ f$. We are done. □

Corollary 3.5. *There holds*

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e} = \frac{\partial f^\alpha}{\partial x^i} \cdot f^*\left(\nabla_{\frac{\partial}{\partial y^\alpha}} e\right). \tag{3.4}$$

Proof. It follows from proposition 3.4 that

$$\left(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}\right)(p) = \widehat{\nabla}_{\frac{\partial}{\partial x^i}\big|_p}\widehat{e} = f^*_p\left(\nabla_{f_*\left(\frac{\partial}{\partial x^i}\big|_p\right)} e\right) = \frac{\partial f^\alpha}{\partial x^i}(p) \cdot f^*_p\left(\nabla_{\frac{\partial}{\partial y^\alpha}\big|_{f(p)}} e\right).$$

We are done. □

Remark 3.6. Formula (3.3) gives us an intuitive understanding of pullback connection. However, since in general case, $df\circ X$ can not be regarded as some element in $\Gamma(N, TN)$, We need to build formulas like (3.4) with the help of coordinates.

3.C. **Pullback curvature.**

Definition 3.7. *Let* $f : M \to (N, g_N, \nabla)$ *be a smooth map. The* ***curvature tensor*** $\widehat{R}$ *of the induced connection* $\widehat{\nabla}$ *on the vector bundle* $f^*TN \to M$ *is given by*

$$\widehat{R}(X, Y, s, t) = \widehat{g}\left(\widehat{\nabla}_X\widehat{\nabla}_Y s - \widehat{\nabla}_Y\widehat{\nabla}_X s - \widehat{\nabla}_{[X,Y]}s, t\right).$$

Generally, let $f : M \to N$ be a smooth map, and let $E \to N$ be a vector bundle equipped with a metric g and an affine connection ∇. Then the ***curvature tensor*** *$\widehat{R}$ of the induced connection $\widehat{\nabla}$ on the vector bundle $f^*E \to M$ is given by*

$$\widehat{R}(X,Y,s,t) = \widehat{g}\left(\widehat{\nabla}_X\widehat{\nabla}_Y s - \widehat{\nabla}_Y\widehat{\nabla}_X s - \widehat{\nabla}_{[X,Y]}s, t\right).$$

Proposition 3.8. *Let $f : M \to (N, g_N)$ be a smooth map. Then $\widehat{R} \in \Gamma(M, T^*M \otimes T^*M \otimes f^*TN \otimes f^*TN)$ can be written as*

$$\widehat{R} = \widehat{R}_{ij\gamma\delta} dx^i \otimes dx^j \otimes \widehat{e}^\gamma \otimes \widehat{e}^\delta$$

where

$$\widehat{R}_{ij\gamma\delta} = R_{\alpha\beta\gamma\delta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}$$

and $R_{\alpha\beta\gamma\delta}$ is the component of the curvature of (N, ∇^N, g_N).

Remark 3.9. To be more precisely,

$$\widehat{R}_{ij\gamma\delta} = R_{\alpha\beta\gamma\delta}\circ f \cdot \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}.$$

Proof. By corollary 3.5 we know

$$\begin{aligned}
\widehat{g}\left(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\gamma, \widehat{e}_\delta\right) &= \widehat{g}\left(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\left(\frac{\partial f^\beta}{\partial x^j}\cdot f^*\left(\nabla_{\frac{\partial}{\partial y^\beta}} e_\gamma\right)\right), \widehat{e}_\delta\right) \\
&= \widehat{g}\left(\frac{\partial^2 f^\beta}{\partial x^i \partial x^j}\cdot f^*\left(\nabla_{\frac{\partial}{\partial y^\beta}} e_\gamma\right) + \frac{\partial f^\beta}{\partial x^j}\cdot\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\left(f^*\left(\nabla_{\frac{\partial}{\partial y^\beta}} e_\gamma\right)\right), \widehat{e}_\delta\right) \\
&= \frac{\partial^2 f^\beta}{\partial x^i \partial x^j}\cdot \widehat{g}\left(f^*\left(\nabla_{\frac{\partial}{\partial y^\beta}} e_\gamma\right), \widehat{e}_\delta\right) + \frac{\partial f^\beta}{\partial x^j}\cdot \widehat{g}\left(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\left(f^*\left(\nabla_{\frac{\partial}{\partial y^\beta}} e_\gamma\right)\right), \widehat{e}_\delta\right) \\
&= \frac{\partial^2 f^\beta}{\partial x^i \partial x^j}\cdot \widehat{g}\left(f^*\left(\nabla_{\frac{\partial}{\partial y^\beta}} e_\gamma\right), \widehat{e}_\delta\right) + \frac{\partial f^\beta}{\partial x^j}\frac{\partial f^\alpha}{\partial x^i}\cdot \widehat{g}\left(f^*\left(\nabla_{\frac{\partial}{\partial y^\alpha}}\nabla_{\frac{\partial}{\partial y^\beta}} e_\gamma\right), \widehat{e}_\delta\right)
\end{aligned}$$

It follows that

$$\widehat{R}_{ij\gamma\delta}(p) = \widehat{g}\left(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\gamma, \widehat{e}_\delta\right)(p) - \widehat{g}\left(\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\gamma, \widehat{e}_\delta\right)(p) = \frac{\partial^2 f^\beta}{\partial x^i \partial x^j}(p)\cdot R_{\alpha\beta\gamma\delta}(f(p))$$

in which we use that $\widehat{g} = f^*g$ as we introduced in subsection 3.B. □

Remark 3.10. Similarly, the above conclusions also hold for a general vector bundle E.

Proposition 3.11. *Let $f : M \to N$ be a smooth map, and let $E \to N$ be a vector bundle equipped with a metric g. Then we have*

$$\widehat{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \widehat{e}\right) = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\cdot f^*\left(R\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}, e\right)\right) \tag{3.5}$$

Pointwisely, we have

$$\widehat{R}(u,v,\omega) = f_p^*(R(f_*u, f_*v, \pi(\omega))) \quad \forall u,v \in T_pM \quad \forall \omega \in (f^*E)_p \tag{3.6}$$

where $\pi : f^*E \to E, (p,u) \mapsto u$.

Proof. It follows from remark 3.10 that

$$\begin{aligned}\widehat{g}\left(\widehat{R}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\widehat{s}\right),\widehat{t}\right) &= \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\cdot g\left(R\left(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta},s\right),t\right)\circ f\\ &= \widehat{g}\left(\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\cdot f^*\left(R\left(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta},s\right)\right),\widehat{t}\right)\end{aligned}$$

Then (3.5) follows. Since $R\in\Gamma\left(M,T^*M\otimes T^*M\otimes(f^*E)^*\otimes f^*E\right)$, (3.6) follows. ☐

Corollary 3.12. *There holds*

$$\begin{aligned}\widehat{\nabla}_{\frac{\partial}{\partial x^k}}\left(\widehat{R}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\widehat{e}\right)\right) &= \frac{\partial}{\partial x^k}\left(\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\right)\cdot f^*\left(R\left(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta},e\right)\right)\\ &\quad+\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\frac{\partial f^\gamma}{\partial x^k}\cdot f^*\left(\nabla_{\frac{\partial}{\partial y^\gamma}}\left(R\left(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta},e\right)\right)\right).\end{aligned}\tag{3.7}$$

Proof. It follows from (3.4) and (3.5). ☐

3.D. **Pullback metric and pullback connection for tangent bundle.** Let $f:M\to(N,g_N)$ be a smooth map. We apply subsection 3.B to the tangent bundle TN equipped with the Levi-Civita connection ∇^N. Then we derive the pullback metric $\widehat{g}$ and the pullback connection $\widehat{\nabla}$ on the pullback bundle f^*TN.

Speaking specifically, on local charts (U,x_i) of M and (V,y^α) of N, setting

$$e_\alpha=\frac{\partial}{\partial y^\alpha},\quad \widehat{e}_\alpha=f^*\left(\frac{\partial}{\partial y^\alpha}\right)\quad\text{and}\quad \widehat{e}^\alpha=f^*\left(dy^\alpha\right),$$

then $(\widehat{e}_\alpha)$ forms a local basis of f^*TN. It follows that $\widehat{g}$ is given by

$$\widehat{g}=\widehat{g}_{\alpha\beta}\widehat{e}^\alpha\otimes\widehat{e}^\beta$$

where

$$\widehat{g}_{\alpha\beta}(p)=\widehat{g}\left(\widehat{e}_\alpha,\widehat{e}_\beta\right)(p)=g_{\alpha\beta}\left(f(p)\right),$$

and that $\widehat{\nabla}$ is given by

$$\widehat{\nabla}\widehat{e}_\alpha=\frac{\partial f^\beta}{\partial x^i}\cdot\Gamma^\gamma_{\alpha\beta}\circ f\cdot dx^i\otimes\widehat{e}_\gamma.$$

In general, by corollary 3.5, we have

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\left(h^\alpha\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right)\right)=\frac{\partial h^\alpha}{\partial x^i}\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right)+h^\alpha\frac{\partial f^\beta}{\partial x^i}\cdot\Gamma^\gamma_{\alpha\beta}\circ f\cdot f^*\left(\frac{\partial}{\partial y^\gamma}\right).\tag{3.8}$$

Moreover, the pullback connection is compact with the pullback metric.

Proposition 3.13. *Let ∇^N be the Levi-Civita connection on TN. The induced connection $\widehat{\nabla}$ is compatible with $\widehat{g}$; i.e. for any $X\in\Gamma(M,TM)$, $s,t\in\Gamma(M,f^*TN)$ one has*

$$X\left(\widehat{g}(s,t)\right)=\widehat{g}\left(\widehat{\nabla}_X s,t\right)+\widehat{g}\left(s,\widehat{\nabla}_X t\right).\tag{3.9}$$

Proof. Since $\widehat{\nabla}$ is an affine connection (proposition 3.2), it suffices to show that

$$v\left(\widehat{g}_{\alpha\beta}\right)=\widehat{g}\left(\widehat{\nabla}_v\widehat{e}_\alpha,\widehat{e}_\beta(p)\right)+\widehat{g}\left(\widehat{e}_\alpha(p),\widehat{\nabla}_v\widehat{e}_\beta\right)$$

for all α, β and $v \in T_pM$. Note that by proposition 3.4 we know

$$\widehat{g}\left(\widehat{\nabla}_v \widehat{e}_\alpha, \widehat{e}_\beta(p)\right) = \widehat{g}\left(f_p^*\left(\nabla_{f_* v} e_\alpha\right), \widehat{e}_\beta(p)\right) = g\left(\nabla_{f_* v} e_\alpha, e_\beta\right)(f(p)).$$

Similarly we have

$$\widehat{g}\left(\widehat{e}_\alpha(p), \widehat{\nabla}_v \widehat{e}_\beta\right) = g\left(e_\alpha, \nabla_{f_* v} e_\beta\right)(f(p)).$$

Since $\nabla = \nabla^N$ is the Levi-Civita connection, it follows that

$$\widehat{g}\left(\widehat{\nabla}_v \widehat{e}_\alpha, \widehat{e}_\beta(p)\right) + \widehat{g}\left(\widehat{e}_\alpha(p), \widehat{\nabla}_v \widehat{e}_\beta\right) = (f_* v)(g_{\alpha\beta}) = v\left(\widehat{g}_{\alpha\beta}\right).$$

We are done. □

3.E. **Revisit the global differential; pushforward of vector fields.** Let $f : M \to N$ be a smooth map. In the next we revisit the global differential df.

(1) We have proved in [Xioc] that $df : TM \to TN$ is smooth, and hence

$$\widetilde{df} : TM \to M \times TN, \quad X_p \mapsto (p, df_p(X_p))$$

is smooth.

(2) As we mentioned in subsection 3.A, f^*E is an embedded submanifold of $M \times E$, and hence f^*TN is an embedded submanifold of $M \times TN$.

(3) Note that $\operatorname{im}\left(\widetilde{df}\right) \subset f^*TN$. It follows from point (2) that $\widetilde{df}$ can be regarded as a smooth map from TM to f^*TN.

(4) By definition, it's clear that $\widetilde{df}$ a smooth bundle homomorphism over M. In particular, $\widetilde{df} = dx^i \otimes f_*\left(\frac{\partial}{\partial x^i}\right) \in \Gamma(M, T^*M \otimes f^*TN)$.

Remark 3.14. If there is no misunderstanding, we sometimes identify $\widetilde{df}$ with df.

Using the smooth bundle homomorphism $\widetilde{df} : TM \to f^*TN$, we can define the pushforward of vector fields.

Definition 3.15 (Pushforward of tangent vector fields). *For any $X \in \Gamma(M, TM)$, the* ***pushforward*** *of X is defined as $f_*X := \widetilde{df} \circ X \in \Gamma(M, f^*TN)$.*

Proposition 3.16. *Let $f : M \to N$ be a smooth map. Then*

(1) $f_*(h \cdot X) = h \cdot f_*X$ for any $X \in \Gamma(M, TM)$ and $h \in C^\infty(M)$;

(2) $(f_*X)_p = f_p^*\left(df_p\left(X_p\right)\right)$ for any $X \in \Gamma(M, TM)$;

(3) On charts (U, x^i) and (V, y^α) with $f(U) \subset V$, we have

$$f_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial f^\alpha}{\partial x^i} \cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right). \tag{3.10}$$

Proof. Claim (1) is trivial. For (2) we note that

$$(f_*X)_p = \left(p, df_p\left(X_p\right)\right) = f_p^*\left(df_p\left(X_p\right)\right).$$

Using proposition 3.4, it follows that

$$\left(f_*\left(\frac{\partial}{\partial x^i}\right)\right)_p = f_p^*\left(df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)\right) = f_p^*\left(\frac{\partial f^\alpha}{\partial x^i}(p) \cdot \left.\frac{\partial}{\partial y^\alpha}\right|_{f(p)}\right) = \frac{\partial f^\alpha}{\partial x^i}(p) \cdot \left(f^*\left(\frac{\partial}{\partial y^\alpha}\right)\right)(p).$$

Then (3) follows. □

Remark 3.17. It follows that

$$\widetilde{d}f = \frac{\partial f^\alpha}{\partial x^i} \cdot dx^i \otimes f^*\left(\frac{\partial}{\partial y^\alpha}\right). \tag{3.11}$$

Corollary 3.18. *Let $f : M \to N$ be a smooth map. Then*

$$f_*\left(X^i \frac{\partial}{\partial x^i}\right) = X^i \frac{\partial f^\alpha}{\partial x^i} \cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right). \tag{3.12}$$

Proof. It follows immediately from proposition 3.16. □

3.F. The second fundamental form.

Definition 3.19. *Let $f : (M, g^M, \nabla^M) \to (N, g^N, \nabla^N)$ be a smooth map between two Riemannian manifolds,*[1] *and $\widehat{\nabla}$ be the affine connection on f^*TN induced by (TN, ∇^N, g^N). For any $X, Y \in \Gamma(M, TM)$, we define*

$$B(X,Y) := \widehat{\nabla}_X (f_* Y) - f_*\left(\nabla^M_X Y\right) \in \Gamma(M, f^*TN).$$

*It is called the **second fundamental form** of $f : (M, g^M) \to (N, g^N)$.*

Proposition 3.20. *Let $f : (M, g) \to (N, h)$ be a smooth map and $\widetilde{\nabla}$ be the affine connection on the vector bundle $T^*M \otimes f^*TN$ induced by Levi-Civita connections ∇^M and ∇^N. Then*

$$B = \widetilde{\nabla}\widetilde{d}f$$

where $\widetilde{d}f$ is regarded as a smooth section in $\Gamma(M, T^*M \otimes f^*TN)$.

Proof. By formulas (3.10) and (3.11) we know that

$$\widetilde{d}f = dx^i \otimes f_*\left(\frac{\partial}{\partial x^i}\right) \in \Gamma(M, T^*M \otimes f^*TN).$$

It follows that

$$\begin{aligned}
(\widetilde{\nabla}_X \widetilde{d}f)(Y) &= \left(dx^i \otimes \widehat{\nabla}_X\left(f_*\left(\frac{\partial}{\partial x^i}\right)\right) + (\nabla_X dx^i) \otimes f_*\left(\frac{\partial}{\partial x^i}\right)\right)(Y) \\
&= Y^i \cdot \widehat{\nabla}_X\left(f_*\left(\frac{\partial}{\partial x^i}\right)\right) + XY^i \cdot f_*\left(\frac{\partial}{\partial x^i}\right) - (\nabla_X Y)(dx^i) \cdot f_*\left(\frac{\partial}{\partial x^i}\right) \\
&= \widehat{\nabla}_X\left(Y^i \cdot f_*\left(\frac{\partial}{\partial x^i}\right)\right) - f_*(\nabla_X Y) = \widehat{\nabla}_X(f_* Y) - f_*(\nabla_X Y)
\end{aligned}$$

where $Y^i = dx^i(Y)$. We are done. □

Proposition 3.21. *Let $f : (M, g^M) \to (N, g^N)$ be a smooth map. Then $B \in \Gamma(M, T^*M \otimes T^*M \otimes f^*TN)$ is symmetric, i.e.*

$$B(X,Y) = B(Y,X),$$

and

$$B = \left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma} \circ f - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k}\right) dx^i \otimes dx^j \otimes f^*\left(\frac{\partial}{\partial y^\alpha}\right) \tag{3.13}$$

where Γ^k_{ij} and $\Gamma^\gamma_{\alpha\beta}$ are Christoffel symbolds of g^M and g^N respectively.

[1] ∇^M and ∇^N are Levi-Civita connections.

Proof. Suppose that $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^j\frac{\partial}{\partial x^j}$. By formula (3.12), we know

$$f_*\left(\nabla_X Y\right) = \left(X^i\frac{\partial Y^j}{\partial x^i} + X^iY^j\Gamma^k_{ij}\right)\frac{\partial f^\alpha}{\partial x^k}\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right). \tag{3.14}$$

By formulas (3.8) and (3.12) , we know

$$\widehat{\nabla}_X\left(f_*Y\right) = \left(X^i\frac{\partial}{\partial x^i}\left(Y^j\frac{\partial f^\alpha}{\partial x^j}\right) + X^iY^j\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma}\circ f\right)\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right). \tag{3.15}$$

Then the conclusion follows. □

Corollary 3.22. *For any $X, Y \in \Gamma(M, TM)$, we have*

$$\widehat{\nabla}_X\left(f_*Y\right) - \widehat{\nabla}_Y\left(f_*X\right) = f_*\left(\nabla_X Y\right) - f_*\left(\nabla_Y X\right) = f_*\left([X,Y]\right). \tag{3.16}$$

Proof. The first equation is equivalent to that B is symmetric, and the second equation just uses the property of Levi-Civita connection. □

Corollary 3.23. *Let $\alpha(u^1, u^2, u^3) : I_1 \times I_2 \times I_3 \to (M, g)$ be a smooth map, where the I_k's are intervals, and let $\widehat{\nabla}$ be the pullback connection of the Levi-Civita connection on M. Then*

$$\widehat{\nabla}_{\frac{\partial}{\partial u^i}}\alpha_*\left(\frac{\partial}{\partial u^j}\right) = \widehat{\nabla}_{\frac{\partial}{\partial u^j}}\alpha_*\left(\frac{\partial}{\partial u^i}\right) \quad \forall i, j.$$

Definition 3.24. *Given a smooth map $f : (M, g) \to (N, h)$, the* ***Laplacian*** *is given by $\Delta_{g,h}f = \mathrm{tr}_g B \in \Gamma(M, f^*TN)$.*

Proposition 3.25. *If $f : (M, g) \to (N, h)$ is a smooth map, then*

$$\left(\Delta_{g,h}f\right)^\gamma = g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j} + \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}(\Gamma_h)^\alpha_{\beta\gamma}\circ f - (\Gamma_g)^k_{ij}\frac{\partial f^\alpha}{\partial x^k}\right) \tag{3.17}$$

If $f : (M, g) \to (N, h)$ is, in addition, a diffeomorphism, then

$$\left(\Delta_{g,h}f\right)^\gamma = -\tilde{g}^{\alpha\beta}\left[(\Gamma_{\tilde{g}})^\gamma_{\alpha\beta}\circ f - (\Gamma_h)^\gamma_{\alpha\beta}\circ f\right] \quad \text{where} \quad \tilde{g} = (f^{-1})^*g. \tag{3.18}$$

Proof. To be continued. (Exercise.) □

3.G. **Isometric immersions.** Let $\left(\overline{M}, \overline{g}, \overline{\nabla}\right)$ be a Riemannian manifold, and let $f : M \to \overline{M}$ be an immersion. Basically, we know:

(1) By subsection 3.B, $\left(T\overline{M}, \overline{g}, \overline{\nabla}\right)$ induces $\left(f^*TM, \widehat{g}, \widehat{\nabla}\right)$;

(2) $\overline{g}$ induces a Riemannian metric $g = f^*\overline{g}$, and g induces a Levi-Civita connection ∇.

Proposition 3.26. *Let $f : M \to \left(\overline{M}, \overline{g}\right)$ be an immersion. Then*

$$g(X, Y) = \widehat{g}(f_*X, f_*Y), \quad X, Y \in \Gamma(M, TM). \tag{3.19}$$

Moreover, under the normal convention, we have

$$g(X, Y) = \widehat{g}(f_*X, f_*Y) = \overline{g}_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}X^iY^j,$$

and

$$g_{ij}(p) = \overline{g}_{\alpha\beta}\left(f(p)\right)\frac{\partial f^\alpha}{\partial x^i}(p)\frac{\partial f^\beta}{\partial x^j}(p).$$

Proof. Trivial. □

In the next we introduce some non-trivial facts, which are based on the fact that the bundle homomorphism $\widetilde{d}f$ over M has constant rank.

Proposition 3.27. *Let $f : M \to (\overline{M}, \overline{g})$ be an immersion. Then the smooth bundle homomorphism $\widetilde{d}f : TM \to f^*T\overline{M}$ over M, which is introduced in subsection 3.E, has constant rank. It follows that*

$$f_*(TM) := \operatorname{im}(\widetilde{d}f) = \bigcup_{p\in M} \operatorname{im}(\widetilde{d}f_p)$$

is a subbundle of $f^*T\overline{M}$. Moreover, there exists a subbundle $T^\perp M$ of $f^*T\overline{M}$ such that

$$f^*T\overline{M} = f_*(TM) \oplus T^\perp M \tag{3.20}$$

where the orthogonal decomposition is with respect to the Riemannian metric $\overline{g}$ on $\overline{M}$.

Proof. Since f is an immersion, clearly $\widetilde{d}f$ has constant rank. By theorem 5.4 we know $f_*(TM)$ is a subbundle of f^*TN. In the next we prove the last assertion.

Set $E = f^*T\overline{M}$ and $F = f_*(TM)$. Then we consider the map

$$\Phi : E \to E, \quad e_p \mapsto \operatorname{Proj}_{F_p}(e_p)$$

where $\operatorname{Proj}_{F_p}$ is the projection map from E_p to F_p with respect to $\widehat{g}_p$. It's easy to see that Φ is a smooth bundle homomorphism over M.[2] Clearly Φ is of constant rank. By theorem 5.4 again, we know that

$$\ker\Phi = \bigcup_{p\in M} F_p^\perp$$

is a subbundle of E. Clearly $\ker\Phi = T^\perp M$ is as required. □

Proposition 3.28. *Let $f : M \to (\overline{M}, \overline{g})$ be an immersion. Using the function Φ in proposition 3.27, then we have*

$$\Phi\left(\widehat{\nabla}_X(f_*Y)\right) = f_*\nabla_X Y, \quad \forall X, Y \in \Gamma(M, TM). \tag{3.21}$$

In particular, (3.21) shows how $\widehat{\nabla}$ induces ∇.

Proof. For convenience, first we set

$$\Phi(X, Y) = \Phi\left(\widehat{\nabla}_X(f_*Y)\right).$$

Since f is an immersion, $f_*(\nabla_X Y)$ induces $\nabla_X Y$. Then by the uniqueness of Levi-Civita connection, proposition 3.16 and formula (3.19), to show (3.21) it suffices to verify that for all $X, Y, Z \in \Gamma(M, TM)$ and $h \in C^\infty(M)$ we have the following points:

(1) $\Phi(hX + Y, Z) = h\Phi(X, Z) + \Phi(Y, Z)$;
(2) $\Phi(X, hY + Z) = (Xh)\cdot f_*Y + h\cdot\Phi(X, Y) + \Phi(X, Z)$;
(3) $X(g(Y, Z)) = \widehat{g}(\Phi(X, Y), f_*Z) + \widehat{g}(f_*Y, \Phi(X, Z))$;
(4) $\Phi(X, Y) - \Phi(Y, X) = f_*([X, Y])$.

[2] This easily follows from the local frame criterion for subbundles and the fact that the rank of a linear map does not decrease under perturbation,

Claims (1) and (2) are trivial, which follow from proposition 3.2 and proposition 3.16. For (3), note that by proposition 3.13, formula (3.19) and proposition 3.27 we have

$$\begin{aligned} X(g(Y,Z)) &= X(\widehat{g}(f_*Y,f_*Z)) = \widehat{g}\left(\widehat{\nabla}_X(f_*Y),f_*Z\right)+\widehat{g}\left(f_*Y,\widehat{\nabla}_X(f_*Z)\right) \\ &= \widehat{g}\left(\Phi\left(\widehat{\nabla}_X(f_*Y)\right),f_*Z\right)+\widehat{g}\left(f_*Y,\Phi\left(\widehat{\nabla}_X(f_*Z)\right)\right) \\ &= \widehat{g}(\Phi(X,Y),f_*Z)+\widehat{g}(f_*Y,\Phi(X,Z)). \end{aligned}$$

For (4), note that by corollary 3.22 and proposition 3.27 we have

$$\Phi(X,Y)-\Phi(Y,X)=\Phi\left(\widehat{\nabla}_X(f_*Y)-\widehat{\nabla}_Y(f_*X)\right)=\Phi(f_*([X,Y]))=f_*([X,Y]).$$

We are done. □

Corollary 3.29. *Let $f:(M,g_M)\to(N,g_N)$ be an* ***isometry****. Then $B=0$.*

Proof. Since f is an isometry, $\dim f_*(TM)=\dim f^*TN$, and hence the map Φ given by proposition 3.27 is the identity map. It follows from corollary 3.28 that $B=0$. □

Corollary 3.30. *Let $f:M\to\left(\overline{M},\overline{g}\right)$ be an immersion. For any $X,Y,Z\in\Gamma(M,TM)$,*

$$\widehat{g}(B(X,Y),f_*Z)=0.$$

In particular, we have

$$B\in\Gamma\left(M,T^*M\otimes T^*M\otimes T^{\perp}M\right)$$

and B is called the ***second fundamental form of the immersion*** *$f:M\to\left(\overline{M},\overline{g}\right)$.*

Proof. It follows from proposition 3.27 and proposition 3.28 that

$$\begin{aligned} \widehat{g}(B(X,Y),f_*Z) &= \widehat{g}\left(\widehat{\nabla}_X(f_*Y)-f_*(\nabla_XY),f_*Z\right) \\ &= \widehat{g}\left(\Phi\left(\widehat{\nabla}_X(f_*Y)\right)-f_*(\nabla_XY),f_*Z\right)=0. \end{aligned}$$

Then the conclusion follows from proposition 3.27. □

Corollary 3.31 (Gauss)**.** *Let $f:M\to\left(\overline{M},\overline{g}\right)$ be an immersion. For any $X,Y,Z,W\in\Gamma(M,TM)$, we have*

$$R(X,Y,Z,W)-\widehat{R}(X,Y,f_*Z,f_*W)=\widehat{g}(B(Y,Z),B(X,W))-\widehat{g}(B(X,Z),B(Y,W)).$$

In particular

$$R(X,Y,Y,X)-\widehat{R}(X,Y,f_*Y,f_*X)=\widehat{g}(B(X,X),B(Y,Y))-\widehat{g}(B(X,Y),B(X,Y)).$$

Proof. Note that

$$\widehat{\nabla}_X\widehat{\nabla}_Y(f_*Z)=\widehat{\nabla}_X(f_*(\nabla_YZ)+B(Y,Z))=f_*(\nabla_X\nabla_YZ)+B(X,\nabla_YZ)+\widehat{\nabla}_X(B(Y,Z))$$

and similar we have

$$\widehat{\nabla}_Y\widehat{\nabla}_X(f_*Z)=f_*(\nabla_Y\nabla_XZ)+B(Y,\nabla_XZ)+\widehat{\nabla}_Y(B(X,Z)).$$

On the other hand, we have

$$\widehat{\nabla}_{[X,Y]}(f_*Z)=f_*\left(\nabla_{[X,Y]}Z\right)+B([X,Y],Z).$$

It follows from corollary 3.30, proposition 3.13 and formula (3.19) that

$$\begin{aligned}
& \widehat{R}(X,Y,f_*Z,f_*W) \\
= \;& R(X,Y,Z,W) + \widehat{g}\left(\widehat{\nabla}_X (B(Y,Z)) + \widehat{\nabla}_Y (B(X,Z)), f_*W\right) \\
= \;& R(X,Y,Z,W) - \widehat{g}\left(B(Y,Z), \widehat{\nabla}_X(f_*W)\right) - \widehat{g}\left(B(X,Z), \widehat{\nabla}_Y(f_*W)\right) \\
= \;& R(X,Y,Z,W) - \widehat{g}(B(Y,Z), B(X,W)) - \widehat{g}(B(X,Z), B(Y,W)).
\end{aligned}$$

We are done. □

3.H. **Revisit isometric immersions via extensions.**

Lemma 3.32. *Let $f : M \to \left(\overline{M}, \overline{g}\right)$ be an immersion. Then for any $\tau \in \Gamma(M, f^*T\overline{M})$ and $p \in M$, there exist $\widetilde{\tau} \in \Gamma\left(\overline{M}, T\overline{M}\right)$ and a neighborhood U of p with*

$$\tau = f^*\widetilde{\tau} \quad on \quad U.$$

*The vector field $\widetilde{\tau}$ is called the **the extension of** τ **on** U.*

Proof. One can refer to [Lee13] lemma 8.6. □

Remark 3.33. In particular, for $X \in \Gamma(M, TM)$, f_*X has an extension $\widetilde{X}$. The vector field $\widetilde{X}$ is also called **the extension of** X **on** U.

In the next we introduce some new properties based on lemma 3.32.

Proposition 3.34. *Let $f : M \to \left(\overline{M}, \overline{g}\right)$ be an immersion, let $X \in \Gamma(M, TM)$ and $\tau \in \Gamma(M, f^*T\overline{M})$, and let $\widetilde{X}$, $\widetilde{\tau}$ be extensions of X, τ on U respectively. Then*

$$\widehat{\nabla}_X \tau = f^*\left(\nabla_{\widetilde{X}}\widetilde{\tau}\right) \quad on \quad U. \tag{3.22}$$

Proof. Proposition 3.4 yields

$$\widehat{\nabla}_{X_p}\tau = \widehat{\nabla}_{X_p}\left(f^*\widetilde{\tau}\right) = f_p^*\left(\nabla_{\widetilde{X}_{f(p)}}\widetilde{\tau}\right) = f^*\left(\nabla_{\widetilde{X}}\widetilde{\tau}\right)\Big|_p$$

Then the conclusion follows. □

Remark 3.35. If we denote the projection from $T_{f(p)}\overline{M}$ to $df_p(T_pM)$ with respect to $\overline{g}_{f(p)}$ by $\widetilde{\Phi}$, then proposition 3.28 and proposition 3.34 yield

$$\widetilde{\Phi}\left(f^*\left(\nabla_{\widetilde{X}}\widetilde{Y}\right)\Big|_p\right) = df_p\left(\nabla_{X_p}Y\right). \tag{3.23}$$

The following proposition shows that extensions gives us a good perspective to deal with the curvatures. (Don't confuse it with Gauss formula 3.31.)

Proposition 3.36. *Let $f : M \to \left(\overline{M}, \overline{g}\right)$ be an immersion, let $X, Y \in \Gamma(M, TM)$ and $\tau \in \Gamma(M, f^*T\overline{M})$, and let $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{\tau}$ be extensions of X, Y, τ on U respectively. Then*

$$\widehat{R}(X,Y,\tau) = f^*\left(R\left(\widetilde{X},\widetilde{Y},\widetilde{\tau}\right)\right) \quad on \quad U. \tag{3.24}$$

Proof. Formula (3.6) yields that

$$\begin{aligned}
\widehat{R}(X,Y,\tau)\Big|_p &= f_p^*\left(R\left(df_p(X_p), df_p(Y_p), \pi(\tau_p)\right)\right) \\
&= f_p^*\left(R\left(\widetilde{X}_{f(p)}, \widetilde{Y}_{f(p)}, \widetilde{\tau}_{f(p)}\right)\right) = f^*\left(R\left(\widetilde{X},\widetilde{Y},\widetilde{\tau}\right)\right)\Big|_p.
\end{aligned}$$

We are done. □

3.I. **Regular surfaces.** Let $S \subset \mathbb{R}^3$ be a regular surface, and let $\phi : U \to D \subset \mathbb{R}^2$ be a local trivialization of S. Then we have the immersion

$$\gamma = i \circ \phi^{-1} : D \xrightarrow{\phi^{-1}} S \xrightarrow{i} \mathbb{R}^3.$$

Let $\overline{g} = g_0 = \delta_{\alpha\beta} dy^\alpha \otimes dy^\beta$ be the standard metric on $\mathbb{R}^3$. Let $u^1 = u$ and $u^2 = v$ be the coordinates on D. Then we have

$$\widetilde{d}\gamma = \frac{\partial \gamma^\alpha}{\partial u^i} \cdot du^i \otimes \gamma^* \left(\frac{\partial}{\partial y^\alpha} \right),$$

and the induced Riemannian metric $g_D = \gamma^* \overline{g}$ satisfies

$$g_D = \delta_{\alpha\beta} \frac{\partial \gamma^\alpha}{\partial u^i} \frac{\partial \gamma^\beta}{\partial u^j} du^i \otimes du^j.$$

The induced metric g_D is also called **the first fundamental form** of the surface S.

Moreover, by proposition 3.27, there exists an orthogonal decomposition

$$\gamma^* T\mathbb{R}^3 = \gamma_*(TD) \oplus T^\perp D.$$

By choosing sufficiently small U if necessary, we assume that $T^\perp D$ is trivial. Let $\mathbf{n}$ be a unit section of $T^\perp D$, and we set

$$B_n(X, Y) = \widehat{g}(B(X, Y), \mathbf{n})$$

where $\widehat{g}$ is the pullback metric. Then $B_n \in \Gamma(M, T^*M \otimes T^*M)$ is the second fundamental form along $\mathbf{n}$ of the surface S.

Proposition 3.37. *There holds*

$$(B_n)_{ij} := B_n \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = \widehat{g} \left(\frac{\partial^2 \gamma^\alpha}{\partial u^i \partial u^j} \cdot \gamma^* \left(\frac{\partial}{\partial y^\alpha} \right), \mathbf{n} \right). \tag{3.25}$$

Proof. Since $\left(\overline{M}, \overline{g}\right) = \left(\mathbb{R}^3, g_0\right)$, it follows from proposition 3.21 that

$$B_n \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = \widehat{g} \left(\frac{\partial^2 \gamma^\alpha}{\partial x^i \partial x^j} \cdot \gamma^* \left(\frac{\partial}{\partial y^\alpha} \right) - \Gamma^k_{ij} \frac{\partial \gamma^\alpha}{\partial x^k} \cdot \gamma^* \left(\frac{\partial}{\partial y^\alpha} \right), \mathbf{n} \right)$$

By corollary 3.18 we know

$$\Gamma^k_{ij} \frac{\partial \gamma^\alpha}{\partial x^k} \cdot \gamma^* \left(\frac{\partial}{\partial y^\alpha} \right) \in \gamma_* (TD).$$

Then the conclusion follows from proposition 3.27. □

Remark 3.38. That is, with the conventions in the classical differential geometry,

$$L = \langle \vec{r}_{uu}, \vec{n} \rangle, \quad M = \langle \vec{r}_{uv}, \vec{n} \rangle, \quad N = \langle \vec{r}_{vv}, \vec{n} \rangle.$$

Moreover, we have the Gauss's Theorema Egregium.

Theorem 3.39 (Gauss's Theorema Egregium)**.** *The Gauss curvature defined as*

$$K = \frac{\det II}{\det I} \tag{3.26}$$

is the sectional curvature of (D, g_D)*, i.e.*

$$K = \frac{R(X,Y,Y,X)}{|X|^2_{g_D}|Y|^2_{g_D} - |g_D(X,Y)|^2}$$

for any linear independent vectors $X, Y \in \Gamma(D, TD)$*.*

Proof. It follows from Gauss formula 3.31 that

$$\begin{aligned} R(X,Y,Y,X) &= \widehat{g}(B(X,X),B(Y,Y)) - \widehat{g}(B(X,Y),B(X,Y)) \\ &= \widehat{g}(B(X,X),\mathbf{n})\cdot\widehat{g}(B(Y,Y),\mathbf{n}) - \widehat{g}(B(X,Y),\mathbf{n})\cdot\widehat{g}(B(X,Y),\mathbf{n}) \\ &= B_n(X,X)\cdot B_n(Y,Y) - B_n(X,Y)\cdot B_n(X,Y). \end{aligned}$$

Set $X = \frac{\partial}{\partial u^1}$ and $Y = \frac{\partial}{\partial u^2}$. Then the conclusion follows. □

3.J. **Summary of formulas.** For pullback bundle f^*E, we have

$$\begin{aligned} \widehat{\nabla}_{\frac{\partial}{\partial x^i}}\left(h^A \widehat{e}_A\right) &= \frac{\partial h^A}{\partial x^i}\cdot\widehat{e}_A + h^A\frac{\partial f^\alpha}{\partial x^i}\cdot\Gamma^B_{\alpha A}\circ f\cdot\widehat{e}_B, \\ \widehat{\nabla}_v\widehat{e} &= f_x^*\left(\nabla_{f_* v}e\right), \\ \widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e} &= \frac{\partial f^\alpha}{\partial x^i}\cdot f^*\left(\nabla_{\frac{\partial}{\partial y^\alpha}}e\right) \\ \widehat{R}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\widehat{e}\right) &= \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\cdot f^*\left(R\left(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta},e\right)\right) \\ \widehat{R}(u,v,\omega) &= f_p^*\left(R\left(f_*u, f_*v, \pi(\omega)\right)\right). \end{aligned}$$

In particular, for $E = TN$ we have

$$\begin{aligned} \widehat{\nabla}_{\frac{\partial}{\partial x^i}}\left(h^\alpha\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right)\right) &= \frac{\partial h^\alpha}{\partial x^i}\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right) + h^\alpha\frac{\partial f^\beta}{\partial x^i}\cdot\Gamma^\gamma_{\alpha\beta}\circ f\cdot f^*\left(\frac{\partial}{\partial y^\gamma}\right), \\ X\left(\widehat{g}(s,t)\right) &= \widehat{g}\left(\widehat{\nabla}_X s, t\right) + \widehat{g}\left(s, \widehat{\nabla}_X t\right), \\ \widetilde{d}f &= \frac{\partial f^\alpha}{\partial x^i}\cdot dx^i\otimes f^*\left(\frac{\partial}{\partial y^\alpha}\right), \\ f_*\left(X^i\frac{\partial}{\partial x^i}\right) &= X^i\frac{\partial f^\alpha}{\partial x^i}\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right), \\ \widehat{\nabla}_X\left(f_*Y\right) &= \left(X^i\frac{\partial}{\partial x^i}\left(Y^j\frac{\partial f^\alpha}{\partial x^j}\right) + X^iY^j\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma}\circ f\right)\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right), \\ f_*\left(\nabla_X Y\right) &= \left(X^i\frac{\partial Y^j}{\partial x^i} + X^iY^j\Gamma^k_{ij}\right)\frac{\partial f^\alpha}{\partial x^k}\cdot f^*\left(\frac{\partial}{\partial y^\alpha}\right), \\ B = \widehat{\nabla}_X\left(f_*Y\right) - f_*\left(\nabla_X Y\right) &= \left(\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j} + \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma}\circ f - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k}\right)dx^i\otimes dx^j\otimes f^*\left(\frac{\partial}{\partial y^\alpha}\right), \\ \widehat{\nabla}_X\left(f_*Y\right) - \widehat{\nabla}_Y\left(f_*X\right) &= f_*\left(\nabla_X Y\right) - f_*\left(\nabla_Y X\right) = f_*\left([X,Y]\right). \end{aligned}$$

For isometric immersions, we have

$$\begin{aligned} g(X,Y) &= \widehat{g}(f_*X, f_*Y), \\ f^*T\overline{M} &= f_*(TM)\oplus T^{\perp}M, \\ \Phi\left(\widehat{\nabla}_X(f_*Y)\right) &= f_*\nabla_X Y, \\ R(X,Y,Z,W)-\widehat{R}(X,Y,f_*Z,f_*W) &= \widehat{g}(B(Y,Z),B(X,W))-\widehat{g}(B(X,Z),B(Y,W)), \end{aligned}$$

and via extensions we have

$$\begin{aligned} \widehat{\nabla}_X\tau &= f^*(\nabla_{\widetilde{X}}\widetilde{\tau}) \quad \text{on} \quad U \\ \widetilde{\Phi}\left(f^*(\nabla_{\widetilde{X}}\widetilde{Y})\Big|_p\right) &= df_p\left(\nabla_{X_p}Y\right) \\ \widehat{R}(X,Y,\tau) &= f^*\left(R\left(\widetilde{X},\widetilde{Y},\widetilde{\tau}\right)\right) \quad \text{on} \quad U. \end{aligned}$$

For regular surfaces, we have

$$\begin{aligned} B_n\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right) &= \widehat{g}\left(\frac{\partial^2\gamma^\alpha}{\partial u^i\partial u^j}\cdot\gamma^*\left(\frac{\partial}{\partial y^\alpha}\right),\mathbf{n}\right), \\ K &= \frac{\det II}{\det I}. \end{aligned}$$

One can look for the numbered formulas in section 3 to get their details.

4. The exponential map of Riemannian manifolds

5. Appendix

5.A. Subbundles.

This subsection is copyed from [Xioc].

Definition 5.1 (Bundle homomorphism). *If $\pi : E \to M$ and $\pi' : E' \to M'$ are vector bundles, a continuous map $F : E \to E'$ is called a* ***bundle homomorphism*** *if there exists a map $f : M \to M'$ satisfying $\pi' \circ F = f \circ \pi$ with the property that for each $p \in M$, the restricted map $F|_{E_p} : E_p \to E'_{f(p)}$ is linear.*

The relationship between F and f is expressed by saying that F ***covers*** *f.*

Remark 5.2. Usually, all maps are assumed to be smooth.

Definition 5.3 (Bundle homomorphism over M). *In the special case in which both E and E' are over the same base space M, a bundle homomorphism covering the identity map of M is called a* ***bundle homomorphism over*** *M.*

Theorem 5.4. *Let E and E' be smooth vector bundles over a smooth manifold M, and let $F : E \to E'$ be a* ***smooth bundle homomorphism over*** *M. Define subsets $\ker F \subset E$ and $\mathrm{im} F \subset E'$ by*

$$\ker F = \bigcup_{p \in M} \ker(F|_{E_p}), \quad \mathrm{im} F = \bigcup_{p \in M} \mathrm{im}(F|_{E_p}).$$

Then $\ker F$ and $\mathrm{im} F$ are smooth subbundles of E and E', respectively, if and only if F has ***constant rank****.*[3]

Proof. Clearly we only need to show the sufficiency. Then the conclusion easily follows from the **local frame criterion for subbundles** ([Lee13] lemma 10.32) and the fact that the rank of a linear map does not decrease under perturbation. One can refer to [Lee13] theorem 10.34 for details. □

5.B. Linear algebra.

Proposition 5.5. *For $A \in M^{n \times n}$ and ε sufficiently small, there holds*

$$(I + \varepsilon A)^{-1} = I - \varepsilon A + O(\varepsilon^2).$$

Proof. Just use the Taylor expansion of $B \mapsto B^{-1}$ near the point $B = I$. □

Lemma 5.6. *For a complex matrix $A \in M^{n \times n}$, we have*

$$\det(\lambda I - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) = \lambda^n - s_1(A) \cdot \lambda^{n-1} + \cdots + (-1)^n s_n(A).$$

where $s_1(A) = \mathrm{tr}(A)$ and $s_n(A) = \det A$. Moreover,

$$s_k(cA) = c^k s_k(A), \quad \forall c \in \mathbb{C}.$$

Proof. Trivial. □

Proposition 5.7. *For $A \in \mathrm{GL}(n, \mathbb{R})$ and ε sufficiently small, there holds*

$$\det(A + \varepsilon B) = \det(A) \cdot \left(1 + \mathrm{tr}\left(A^{-1}B\right)\varepsilon + O(\varepsilon^2)\right).$$

[3] For each $p \in M$, the rank of the linear map $F|_{E_p}$ is called the **rank of** F **at** p. We say that F has **constant rank** is its rank is the same for all $p \in M$.

Proof. Note that

$$\det(A + \varepsilon B) = \det(A) \cdot \det\left(I + \varepsilon A^{-1}B\right) = \det(A) \cdot \varepsilon^n \cdot \det\left(\frac{1}{\varepsilon}I - \left(-A^{-1}B\right)\right)$$

Then the conclusion follows from lemma 5.6. □

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